q-Series and Modularity

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Contents

Abstract

In this thesis, we construct two-parameter generalisations of Hecke-Appell type expansions for the generating functions of unimodal and special unimodal sequences. We obtain their explicit representations in terms of mixed false theta series. We use these representations to recover partial theta identities from Ramanujan's lost notebook and in work of Warnaar.

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Thank you.

Chapter 1

Introduction

A modular form is an analytic object with intrinsic symmetric properties. Over the past two hundred years, the study of modular forms has enjoyed fruitful interdependencies with many areas such as number theory, algebraic geometry, combinatorics and mathematical physics. In particular, they were the key players in Maryna Viazovska's spectacular result on the sphere packing problem [45] for which she was awarded the 2022 Fields Medal. She studied packings of lattice points by defining theta functions associated to integer lattices in dimensions 8 and 24. Modular properties of combinatorial generating functions often help determine asymptotics or congruences for the associated coefficients. Modular forms also played an integral part in Andrew Wiles' proof [48] of Fermat's last theorem, relating them to elliptic curves. More concretely, a *modular form* f of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ is a holomorphic function on the complex upper-half plane $\mathcal{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$ satisfying

- (Transformation Condition):
  $\forall \tau \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$, and
- (Growth Condition): f is holomorphic at infinity.

This definition can be extended to include half-integer weights and congruence subgroups, and can be generalised in various ways. For further details, see [19]. Prototypical examples of modular forms are specialisations of the *theta function*

$$\Theta(x; q) := (x; q)_\infty (q/x; q)_\infty (q; q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n q^{\binom{n}{2}} x^n \tag{1.0.1}$$

where $x \in \mathbb{C}^\times$ and $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$, and the last equality is due to the Jacobi triple-product identity. For instance,

$$\Theta(-q^{\frac{1}{2}};q) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} \tag{1.0.2}$$

is a modular form of weight $\frac{1}{2}$. Here and throughout, we use the standard q-Pochhammer symbol

$$(a)_n = (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$.

Roughly speaking, a *q-series* (or *q-hypergeometric series*) is any convergent power series in q assembled from $(a)_n$. For example, one can find

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \tag{1.0.3}$$

in Ramanujan's last letter to G. H. Hardy on the 12^{th} of January, 1920. The theory of q-series began with the famous partition theoretic theorem of Euler in 1750, was systematically developed by Heine in 1847, and was further expanded by F. H. Jackson and L. J. Rogers at the end of the 19th and the beginning of the 20th centuries. Three classes of q-series which eluded classification as modular forms are *mock theta functions*, *partial theta series* and *false theta functions*, all of which were studied by Ramanujan.

The work of Hardy and Ramanujan is one of the most famous collaborations in the history of mathematics. Unfortunately, Ramanujan died at the age of 32, within a year of moving back to India from England. Ramanujan sent only one letter to Hardy in this time before he died. In this last letter he presented 17 functions which he claimed behaved like modular forms but did not transform like them. He called them mock theta functions. He presented them in four groups: one of order 3, two order 5, and one of order 7, as well as equations relating functions within each group. These relations became known as the *mock theta conjectures.* In 2002, Zwegers [51] made the groundbreaking step in understanding how Ramanujan's mock theta functions, e.g., (1.0.3) fit into the theory of modular forms. For example, one can show that [25]

$$f(q) = 2m(-q,q^3,q) + 2m(-q,q^3,q^2).$$

Here, for $x, z \in \mathbb{C}^\times$ and $|q| < 1$ such that neither z nor xz is an integer power of q, we define the *Appell-Lerch sum* as

$$m(x,q,z) := \frac{1}{\Theta(z;q)} \sum_{r\in\mathbb{Z}} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1}xz}. \tag{1.0.4}$$

The key is to realise that there is a precise relationship between mock theta functions and harmonic weak Maass forms [18]. These latter functions have a Fourier expansion with coefficients that can be classified into a "non-holomorphic part" and a "holomorphic part". Following Zagier [49], the "holomorphic part" is called a *mock modular form*. In [51], Zwegers proves that appropriate specialisations of (1.0.4) are weight $\frac{1}{2}$ mock modular forms. All of the classical examples of mock theta functions due to Ramanujan, Watson and others fit into this picture as they are holomorphic parts of harmonic weak Maass forms [18]. This realisation has many astonishing consequences and further developments concerning mock modular forms would have striking implications not only in mathematics, but also in, e.g., wall-crossing phenomena in the theory of black holes [22], the dynamics of supersymmetric field theories [31], homological mirror symmetry [39] and conformal field theory [43]. For a superb overview of the history, theory and applications of mock modular forms, see [23, 49].

In 1923, the University of Madras sent Hardy a packet of Ramanujan's notes and papers which (according to Bruce Berndt [11]) most likely contained what is now called "the lost notebook". These notes were in possession of Watson when he died and were nearly incinerated alongside the other "clutter" in his office, but were retrieved by Rankin and sent to Trinity College Cambridge in 1968. They sat there for 7.5 years before Slater suggested to Andrews to sort through them while visiting. During his visit, Andrews found this sheaf of 158 pages which featured the mock theta conjectures and many other unproven identities. Andrews and Berndt compiled Ramanujan's notes and provided full proofs as part of the five volume series [5–9], the last of which was published in 2018. Ramanujan's work has inspired over a century of mathematics despite gathering dust (and nearly being incinerated...) for more than 40 years.

Importantly, the lost notebook contained a number of identities involving *partial theta series*, i.e., specialisations of sums of the form

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} x^n, \text{ e.g., } \sum_{n=0}^{\infty} q^{\frac{n^2}{2}}. \tag{1.0.5}$$

Note that the sum in (1.0.5) is over $\{n \in \mathbb{Z} : n \geq 0\}$ whereas the sum in (1.0.1) is over $\mathbb{Z}$. As seen in [2,3], [6, Entry 6.3.2], [46, p. 379], [6, Entry 6.3.7] and [6, Entry 6.3.4], we have the following four partial theta identities.

$$\sum_{n\geq 0} \frac{q^n}{(-xq)_n(-q/x)_n} = (1+x)\sum_{n\geq 0} x^{3n} q^{\frac{n(3n+1)}{2}}(1-x^2q^{2n+1}) - \frac{(1+x)(q)_\infty}{\Theta(-x;q)}\sum_{n\geq 0}(-1)^n x^{2n+1} q^{\frac{n(n+1)}{2}}, \tag{1.0.6}$$

$$\sum_{n\geq 0} \frac{q^{2n}}{(xq)_n(q/x)_n} = (1-x)\left(1+x+(1+x^2)\sum_{n\geq 1}(-1)^n x^{3n-2} q^{\frac{n(3n-1)}{2}}(1+xq^n) + \frac{x^2+(1+x^2)\sum_{n\geq 1}(-1)^n x^{2n} q^{\binom{n+1}{2}}}{(x)_\infty(q/x)_\infty}\right), \tag{1.0.7}$$

$$\sum_{n\geq 0} \frac{(-q)_{2n}q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} = \left(\frac{x}{x+1}\right)\left(-\sum_{n\geq 0}(-x)^n q^{n(n+1)} + \frac{\Theta(-q;q^4)}{\Theta(xq;q^2)}\sum_{n\geq 0}(-x)^n q^{\frac{n(n+1)}{2}}\right) \tag{1.0.8}$$

and

$$\sum_{n\geq 0} \frac{q^{2n+1}}{(-xq;q^2)_{n+1}(-q/x;q^2)_{n+1}} = \sum_{n\geq 0} x^{3n+1} q^{3n^2+2n}(1-xq^{2n+1}) - \frac{(q^2;q^2)_\infty}{\Theta(-xq;q^2)}\sum_{n\geq 0}(-1)^n x^{2n+1} q^{n(n+1)}. \tag{1.0.9}$$

For example, one can find the identity (1.0.6) on lines 7, 8 and 9 of page 37 of the lost notebook [40] (see Figure 1.1).

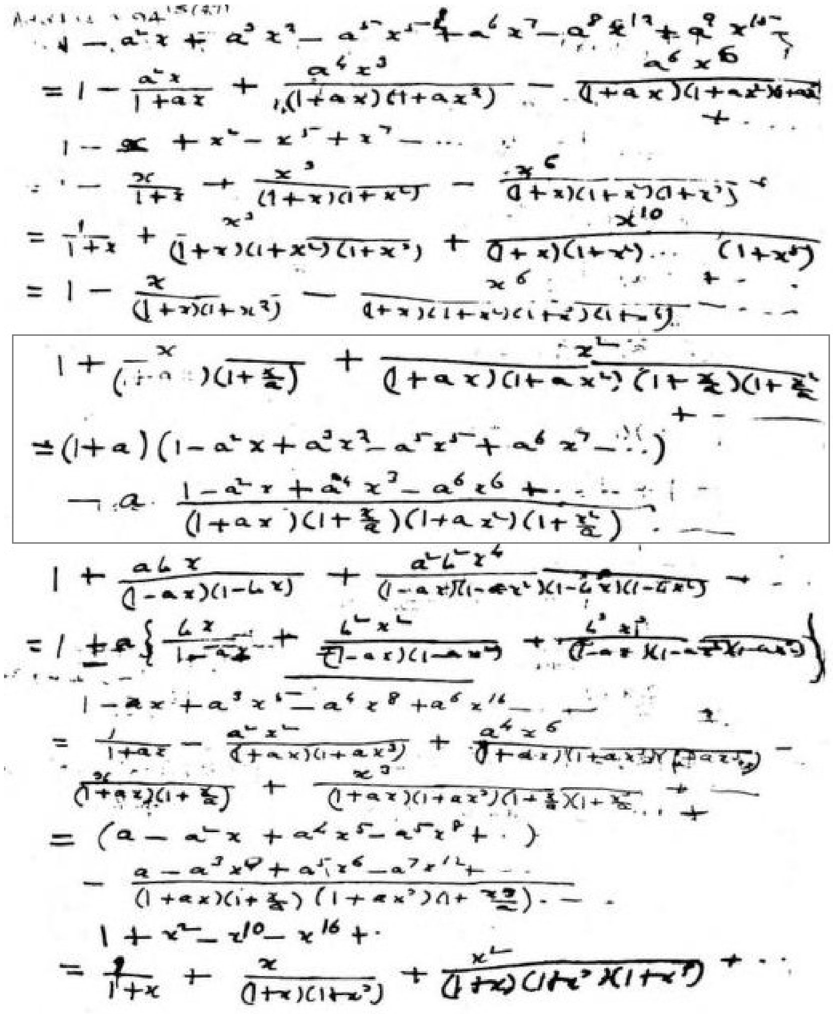

Figure 1.1: Identity (1.0.6) on page 37 of Ramanujan's lost notebook

False theta functions originate in the work of L. J. Rogers [42] and are similar to classical theta functions but are not modular forms as they contain "sign flips". Throughout this thesis, we say that a specialisation of a q-series of the form

$$\sum_{n\in\mathbb{Z}} \mathrm{sg}(n)(-1)^n q^{\binom{n}{2}} x^n$$

is a *false theta series* where

$$\mathrm{sg}(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ -1 & \text{if } r < 0. \end{cases} \tag{1.0.10}$$

False theta series have also been extensively studied from the perspective of q-series and combinatorics. In particular, they occur in q-series identities in Ramanujan's lost notebook. In [5, Chapter 9], one finds the following identities involving false theta series

$$\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}} = \sum_{n\in\mathbb{Z}} \mathrm{sg}(n) q^{2n^2+n} \tag{1.0.11}$$

and

$$\sum_{n\geq 0} q^{3\binom{n+1}{2}-n}(1-q^{2n+1}) = \sum_{n\in\mathbb{Z}} \mathrm{sg}(n) q^{3\binom{n+1}{2}-n}. \tag{1.0.12}$$

Specialising (1.0.6) also yields examples of false theta functions, e.g., when $a = 1$,

$$\sum_{n\geq 0}\frac{q^n}{(-q)_n^2} = 4\sum_{n\in\mathbb{Z}} \mathrm{sg}(n) q^{3\binom{n+1}{2}} - \frac{(q)_\infty}{\Theta(-q;q)}\sum_{n\in\mathbb{Z}} \mathrm{sg}(n) q^{2n^2+n} \tag{1.0.13}$$

and when $a = -1$,

$$\sum_{n\geq 0}\frac{q^n}{(q)_n^2} = \frac{1}{(q)_\infty^2}\sum_{n\in\mathbb{Z}} sg(n) q^{2n^2+n}. \tag{1.0.14}$$

For completeness, we summarise the notation introduced so far by compiling examples of the key building blocks for each modular-like function.

Modular form:	Mock modular form:	False theta series:	Partial theta series:
$\Theta(-q^3;q^4)$	$m(q,q^4,-q^3)$	$\sum_{n\in\mathbb{Z}} \mathrm{sg}(n) q^{2n^2+n}$	$\sum_{n\geq 0} q^{2n^2+n}$

The starting point in this thesis lies in the study of modular properties for combinatorial q-series, namely the generating function of strongly unimodal sequences.

1.1 Strongly Unimodal Sequences

A sequence of positive integers is *strongly unimodal* if

$$a_1 < \ldots < a_r < c > b_1 > \ldots > b_s \tag{1.1.1}$$

with $n = c + \sum_{j=1}^r a_j + \sum_{j=1}^s b_j$. Here, c is the *peak* and n is the *weight* of the sequence. The *rank* of such a sequence is defined as $s - r$, i.e., the number of terms after c minus the number of terms before c. For example, there are six strongly unimodal sequences of weight 5, namely

$$(5),\ (1,4),\ (4,1),\ (1,3,1),\ (2,3),\ (3,2).$$

The ranks are 0, -1, 1, 0, -1 and 1, respectively. Let $u(m,n)$ be the number of such sequences of weight n and rank m. Note that

$$(-xq)_n = \sum_{0 \le i,j \le \frac{n(n+1)}{2}} p(i,j) x^i q^j$$

is a (terminating) generating function with coefficients $p(i,j)$ counting the number of partitions of j with i distinct parts. We can construct a strongly unimodal sequence of weight n, with peak c, by concatenating a (left) partition of size less than c with distinct parts, a (peak) part of size exactly c and a (right) partition of size less than c with distinct parts, and count them accordingly (see Figure 1.2) to obtain the generating function

$$\begin{aligned} U(x;q) := \sum_{\substack{n \ge 1 \\ m \in \mathbb{Z}}} u(m,n) x^m q^n &= \sum_{c \ge 1} (-xq)_{c-1} (-x^{-1}q)_{c-1} q^c \\ &= \sum_{n \ge 0} (-xq)_n (-x^{-1}q)_n q^{n+1}. \end{aligned}$$

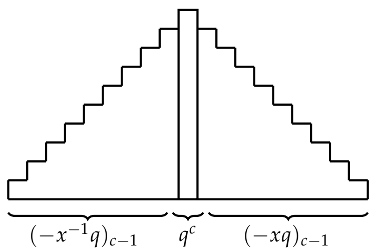

Figure 1.2: Construction of a strongly unimodal sequence.

Such sequences are not only abundant in algebra, combinatorics and geometry [13, 14, 44], but have recent intriguing connections to knot theory and modular forms [20, 34].

In 2015, Hikami and Lovejoy [26] introduced the generalised U-function

$$U_t^{(m)}(x;q) := q^{-t} \sum_{\substack{k_t \geq \cdots \geq k_1 \geq 0 \\ k_m \geq 1}} (-xq)_{k_t-1}(-x^{-1}q)_{k_t-1} q^{k_t} \times \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} - k_i - i + \sum_{j=1}^{i}(2k_j + \chi(m > j)) \\ k_{i+1} - k_i \end{bmatrix} \tag{1.1.2}$$

where $t, m \in \mathbb{Z}$ with $1 \leq m \leq t$, $\chi(X) := 1$ if X is true and $\chi(X) := 0$ otherwise and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the standard q-binomial coefficient. Note that

$$U_1^{(1)}(x;q) = q^{-1}U(x;q).$$

The motivation for (1.1.2) arises in quantum topology. Let K be a knot and $J_N(K;q)$ be the Nth coloured Jones polynomial, normalised to be 1 for the unknot. By computing an explicit formula for the cyclotomic coefficients of the coloured Jones polynomial of the left-handed torus knots $T^*_{(2,2t+1)}$ [26, Proposition 3.2] and comparing with (1.1.2), one obtains the following relation between unimodal sequences and torus knots: Thus,

$$U_t^{(1)}(-q^N;q) = J_N(T^*_{(2,2t+1)};q).$$

$U_t^{(m)}(x;q)$ can be viewed as "extracted" from $J_N(T^*_{(2,2t+1)};q)$. In addition, Hikami and Lovejoy proved the Hecke-Appell type expansion [26, Theorem 5.6]

$$U_t^{(m)}(-x;q) = -q^{-\frac{t}{2}-\frac{m}{2}+\frac{3}{8}} \frac{(qx)_\infty(x^{-1}q)_\infty}{(q)_\infty^2} \times \left(\sum_{\substack{r,s \geq 0 \\ r \not\equiv s \pmod 2}} - \sum_{\substack{r,s < 0 \\ r \not\equiv s \pmod 2}} \right) \frac{(-1)^{\frac{r-s-1}{2}} q^{\frac{1}{8}r^2 + \frac{4t+3}{4}rs + \frac{1}{8}s^2 + \frac{1+m+t}{2}r + \frac{1-m+t}{2}s}}{1 - xq^{\frac{r+s+1}{2}}} \tag{1.1.3}$$

and stated [26, page 13] "... *it is hoped that the Hecke series expansions established in this paper will turn out to be useful for determining modular transformation formulae for* $U_t^{(m)}(x;q)$."

They studied the base case [26, Theorem 4.1] and showed that

$$U_1^{(1)}(-x;q) = \frac{1}{(1-x)(q)_\infty}\Big(\sum_{n,r\geq 0} - \sum_{n,r<0}\Big)(-1)^{n+r}x^{-r}q^{n(3n+5)/2+2nr+r(r+3)/2} \tag{1.1.4}$$

which, by work of Hickerson and Mortenson [25], is a mixed mock modular form. A mixed mock modular form is an expression of the form [22, 33]

$$\sum_{i=1}^{N} h_i g_i$$

where h_i is a modular form and g_i is a mock modular form. We remark that the modular forms need not have equal weight. Naturally, one wonders if the same is true for $U_t^{(m)}(x;q)$. In recent striking work [35], Mortenson and Zwegers show that this is indeed the case by expressing $U_t^{(m)}(x;q)$ in terms of finite sums of Hecke-type double sums

$$f_{a,b,c}(x,y;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}q^{a\binom{r}{2}+brs+c\binom{s}{2}}x^r y^s \tag{1.1.5}$$

where a, b and c are positive integers,

$$\mathrm{sg}(r,s) := \frac{\mathrm{sg}(r)+\mathrm{sg}(s)}{2}. \tag{1.1.6}$$

and $\mathrm{sg}(r)$ is given by (1.0.10). Precisely, they prove for $t \geq 2$ and $1 \leq m \leq t$ [35, Theorem 1.7, Corollary 5.3]

$$\begin{aligned}(1-x)U_{t-1}^{(m)}(-x;q) = \frac{q^{-m+1-t}}{(q)_\infty^3}&\sum_{k=0}^{2t-1}(-1)^k q^{\binom{k+1}{2}} \\ &\times \Big(f_{1,4t-1,1}(q^{k+m+t}, q^{k-t-m+1};q) - q^m f_{1,4t-1,1}(q^{k-t+m+1}, q^{k-m+t};q)\Big) \\ &\times f_{1,2t,2t(2t-1)}(x^{-1}q^{1+k}, -q^{(2t-1)(k+t)+t};q).\end{aligned} \tag{1.1.7}$$

As discussed in Chapter 2, one can show that the expression within the brackets in (1.1.7) is (up to an appropriate power of q) a modular form while the remaining double sum is a mixed mock modular form. Thus, $U_t^{(m)}(x;q)$ is a mixed mock modular form.

1.2 Statement of Results

A sequence of positive integers is *unimodal* if each $<$ is replaced with $\leq$ in (1.1.1), i.e.,

$$1 \leq a_1 \leq \ldots \leq a_r \leq c \geq b_1 \geq \ldots \geq b_s \geq 1. \tag{1.2.1}$$

We write $\bar{c}$ for the distinguished peak as it may not be unique. The rank of such a sequence is again $s - r$. For example, there are 12 unimodal sequences of weight 4, namely

$$\begin{gathered}(\bar{4}),\ (1,\bar{3}),\ (\bar{3},1),(1,\bar{2},1),\ (\bar{2},2),\ (2,\bar{2}),\\ (1,1,\bar{2}),\ (\bar{2},1,1),\ (\bar{1},1,1,1),(1,\bar{1},1,1),\ (1,1,\bar{1},1),\ (1,1,1,\bar{1}).\end{gathered} \tag{1.2.2}$$

These sequences have numerous other guises [4, Section 3] and appear in a wide variety of areas [41].

Inspired by (1.1.3) and (1.1.7), in this thesis we first consider two-parameter generalisations of Hecke-Appell type expansions for the five generating functions of unimodal and special unimodal sequences which appear in [15, 28, 29]. To our knowledge, this covers all known cases of unimodal sequences whose generating function has such an expansion. We then find explicit representations for these generalisations in terms of mixed false theta functions, i.e., expressions of the form

$$\sum_{i=1}^{N} h_i g_i$$

where h_i is a modular form and g_i is a false theta function. These new occurrences of mixed modularity nicely complement (1.1.7) and hint at a general underlying structure for Hecke-Appell type expansions with such properties. This is discussed in Chapter 5. As an application, we demonstrate how the base cases of our results recover the partial theta identities (1.0.6)–(1.0.9). The results in this thesis appeared in [1].

Remark 1.2.1. *Similar to mixed mock modular forms, the modular forms which feature in the summand of a mixed false theta functions need not have the same weight. However, the modular forms in the summands of Theorems 1.2.3, 1.2.5, 1.2.7, 1.2.9 and 1.2.11 all have weight* 1*, e.g., expressions of the form* $f_{1,2t,1}(x, y; q)$ *feature in Theorems 1.2.3, 1.2.5 and 1.2.9. Consider the case when* $t = 1$ *and* x, y *are integer powers of* q*. Using the identity [25]*

$$f_{1,2,1}(x,y;q) = \Theta(y;q)m(\frac{q^2x}{y^2},q^3,-1)+\Theta(x;q)m(\frac{q^2y}{x^2},q^3,-1) - \frac{y(q^3;q^3)_\infty^3\Theta(-x/y;q)\Theta(q^2xy;q^3)}{\Theta(-1;q^3)\Theta(-qy^2/x;q^3)\Theta(-qx^2/y;q^3)},$$

one can see that the first two terms vanish. The weight of the remaining modular form can be computed by noting that $(q^3;q^3)_\infty^3$ *has weight* $\frac{3}{2}$ *and each* Θ*-function has weight* $\frac{1}{2}$*. Analogous computations can be carried out using formulas from [25] and [35].*

1.2.1 Modularity of $\mathcal{U}_t^{(m)}(x;q)$

For the first case, let $\mathbf{u}(m,n)$ denote the number of unimodal sequences of weight n and rank m and consider its generating function [28, Eq. (2.2)]

$$\mathcal{U}(x;q) := \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} \mathbf{u}(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^n}{(xq)_n(q/x)_n} \tag{1.2.3}$$

which satisfies the Hecke-Appell type expansion [28, Eq. (2.5)]

$$\mathcal{U}(x;q) = \frac{(1-x)}{(q)_\infty^2}\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2rs+\frac{s^2}{2}+\frac{3}{2}r+\frac{1}{2}s}}{1-xq^r}. \tag{1.2.4}$$

Remark 1.2.2. *A unimodal sequence can be constructed by fixing a peak of size n, with two partitions of weight at most n on either side, i.e., the unimodal sequences with peak n are counted by*

$$\frac{q^n}{(xq)_n(q/x)_n}$$

where x tracks the number of parts of the partitions to the right of the peak and x^{-1} tracks the number of parts of the partitions on the left. This yields (1.2.3).

For t, $m \in \mathbb{Z}$ with $t \geq 1$, $-t \leq m \leq 3t-2$ and $t \equiv m \pmod 2$, consider the generalisation

$$g_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{t+1+m}{2}r+\frac{t+1-m}{2}s}}{1-xq^r} \tag{1.2.5}$$

and

$$\mathcal{U}_t^{(m)}(x;q) := \frac{(1-x)}{(q)_\infty^2} g_{t,m}(x;q). \tag{1.2.6}$$

By (1.2.4)–(1.2.6), $\mathcal{U}_1^{(1)}(x;q) = \mathcal{U}(x;q)$. Following [25], we use the term “generic” to mean that the parameters do not cause poles in the Appell-Lerch series (1.0.4) or in the quotients of theta functions which occur after applying (2.1.1) to the Hecke-type double sums. Our first result shows that $\mathcal{U}_t^{(m)}(x;q)$ is a mixed false theta function.

Theorem 1.2.3. *Let t, $m \in \mathbb{Z}$ with $t \geq 1$, $-t \leq m \leq 3t-2$ and $t \equiv m \pmod 2$. For generic x, we have*

$$\mathcal{U}_t^{(m)}(x;q) = \frac{(1-x)}{\Theta(x;q)} \frac{q^{1-3t^2-\frac{t-m}{2}+tm}}{(q)_\infty^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{4t^2k-t^2+tm-\frac{t-m}{2}+8t^4}; q). \tag{1.2.7}$$

Hence, $\mathcal{U}_t^{(m)}(x;q)$ is a mixed false theta series.

1.2.2 Modularity of $W_t^{(m)}(x;q)$

For the second case, consider unimodal sequences with a double peak, i.e., sequences of the form

$$a_1 \leq \ldots \leq a_r \leq \overline{c}\,\overline{c} \geq b_1 \geq \ldots \geq b_s$$

with weight $n = 2c + \sum_{i=1}^r a_i + \sum_{i=1}^s b_i$. For example, there are eleven such sequences of weight 6, namely

$$(\overline{3},\overline{3}),\ (\overline{2},\overline{2},2),\ (2,\overline{2},\overline{2}),\ (\overline{2},\overline{2},1,1), (1,\overline{2},\overline{2},1),\ (1,1,\overline{2},\overline{2}),$$
$$(\overline{1},\overline{1},1,1,1,1),\ (1,\overline{1},\overline{1},1,1,1),\ (1,1,\overline{1},\overline{1},1,1), (1,1,1,\overline{1},\overline{1},1),\ (1,1,1,1,\overline{1},\overline{1}).$$

The rank of such a unimodal sequence is $s-r$ where we assume that the empty sequence has rank 0. Let $W(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [29, Eq. (2.1)]

$$W(x;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} W(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^{2n}}{(xq)_n(q/x)_n} \tag{1.2.8}$$

which satisfies the Hecke-Appell type expansion [29, Eq. (2.3)]

$$W(x;q) = \frac{(1-x)}{(q)_\infty^2}\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^{r+s} q^{\frac{r^2}{2}+2rs+\frac{s^2}{2}+\frac{r}{2}+\frac{s}{2}}(1+q^{2r})}{1-xq^r} - \frac{1}{(xq)_\infty(q/x)_\infty}. \tag{1.2.9}$$

Remark 1.2.4. *The derivation of* (1.2.8) *is similar to $\mathcal{U}(x;q)$ but we replace q^n with q^{2n} in* (1.2.3) *because we count the peak twice.*

For $t, m \in \mathbb{Z}$ with $t \geq 1, 1-t \leq m \leq t$, consider the generalisation

$$h_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^{r+s} q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}(1+q^{2mr})}{1-xq^r} \tag{1.2.10}$$

and

$$W_t^{(m)}(x;q) := \frac{(1-x)}{(q)_\infty^2} h_{t,m}(x;q) - \frac{1}{(xq)_\infty (q/x)_\infty}. \tag{1.2.11}$$

By (1.2.9)–(1.2.11), $W_1^{(1)}(x;q) = W(x;q)$. Our second result demonstrates that $W_t^{(m)}(x;q)$ is the sum of a mixed false theta function and a modular form.

Theorem 1.2.5. *Let $t, m \in \mathbb{Z}$ with $t \geq 1, 1-t \leq m \leq t$. For generic x, we have*

$$\begin{aligned} W_t^{(m)}(x;q) = \frac{(1-x)q^{1-m-2t^2}}{(q)_\infty^2 \Theta(x;q)} &\sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\ &\times \Big(f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{8t^4-m+4t^2k}; q) \\ &\qquad + f_{1,4t^2-1,4t^2(4t^2-1)}(xq^{k+1}, -q^{8t^4-m+4t^2k}; q)\Big) \\ &- \frac{1}{(xq)_\infty (q/x)_\infty}. \end{aligned} \tag{1.2.12}$$

Hence, $W_t^{(m)}(x;q)$ is a mixed false theta series.

1.2.3 Modularity of $\mathcal{V}_t^{(m)}(x;q)$

For the third case, consider unimodal sequences where c is odd, $\sum a_i$ is a partition without repeated even parts and $\sum b_i$ is an overpartition into odd parts whose largest part is not $\bar{c}$. For example, there are twelve such sequences of weight 5, namely

$$(\bar{5}),\ (1,\bar{3},1),\ (1,1,\bar{3}),\ (\bar{3},1,1),\ (\bar{3},\bar{1},1),\ (1,\bar{3},\bar{1}), (2,\bar{3}),$$
$$(1,1,1,1,\bar{1}),\ (1,1,1,\bar{1},1),\ (1,1,\bar{1},1,1),\ (1,\bar{1},1,1,1), (\bar{1},1,1,1,1).$$

The rank of such a sequence is the number of odd non-overlined parts in $\sum b_i$ minus the number of odd parts in $\sum a_i$. Let $\mathcal{V}(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [29, Eq. (4.1)]

$$\mathcal{V}(x;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \mathcal{V}(m,n)x^m q^n = \sum_{n\geq 0} \frac{(-q)_{2n}q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \tag{1.2.13}$$

which satisfies the Hecke-Appell type expansion [29, Eq. (4.3)]

$$\mathcal{V}(x;q) = \frac{1}{(q)_\infty(q^2;q^2)_\infty}\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{r^2+2rs+\frac{s^2}{2}+3r+\frac{3s}{2}+1}}{(1+q^{2r+1})(1-xq^{2r+1})}. \tag{1.2.14}$$

Remark 1.2.6. *An overpartition is a partition in which the first occurrence of a part may be overlined [21]. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have*

$$\bar{\mathcal{P}}(q) := \sum_{n\geq 0} \bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}$$

where $\bar{p}(n)$ is the number of overpartitions of n. We obtain (1.2.13) *by first counting the peak with q^{2n+1}. Here, $\sum b_i$ can be expressed by $\frac{(-q;q^2)_{n+1}}{(q;q^2)_{n+1}}$. In order to account for the rank, we need to include an x term to track the non-overlined parts (the denominator) to obtain $\frac{(-q;q^2)_{n+1}}{(xq;q^2)_{n+1}}$. Also, $\sum a_i$ can be interpreted as a sequence consisting of a partition of odd parts of size at most* $2n+1$*, and a partition with distinct even parts of size at most* $2n$*, i.e., $\frac{(-q^2;q^2)_n}{(q;q^2)_{n+1}}$. In order to track the odd parts to the left of the peak, we include an x^{-1} to obtain $\frac{(-q^2;q^2)_n}{(x^{-1}q;q^2)_{n+1}}$. So, such sequences with peak* $2n+1$ *are counted by*

$$\mathcal{V}(x;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \mathcal{V}(m,n)x^m q^n = \sum_{n\geq 0} \frac{(-q;q^2)_{n+1}(-q^2;q^2)_n q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}$$

and the numerator simplifies to yield (1.2.13)*.*

For t, $m \in \mathbb{Z}$ where $t \geq 1$, consider the generalisation

$$\overline{k}_{t,m}(x;q) = \frac{xq}{1+x}\left(\frac{1}{x}k_{t,m}(-1;q) + k_{t,m}(x;q)\right)$$

where

$$k_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{2\binom{r}{2}+2trs+\binom{s}{2}+2(t+1)r+2ms}}{1-xq^{2r+1}} \tag{1.2.15}$$

and

$$\mathcal{V}_t^{(m)}(x;q) := \frac{1}{(q)_\infty (q^2;q^2)_\infty} \overline{k}_{t,m}(x;q). \tag{1.2.16}$$

By (1.2.14)–(1.2.16), $\mathcal{V}_1^{(1)}(x;q) = \mathcal{V}(x;q)$. Our third result establishes that $\mathcal{V}_t^{(m)}(x;q)$ is a mixed false theta function.

Theorem 1.2.7. *Let t, $m \in \mathbb{Z}$ where $t \geq 1$. For generic x, we have*

$$\begin{aligned}\mathcal{V}_t^{(m)}(x;q) = \frac{q^{-2t^2+3t-4tm}}{(1+x)(q)_\infty (q^2;q^2)_\infty} \sum_{k=0}^{2t^2-2} (-1)^k q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\ \times \Bigg(\frac{1}{\Theta(-q;q^2)} f_{1,2t^2-1,2t^2(2t^2-1)}(-q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q^2) \\ - \frac{1}{\Theta(qx;q^2)} f_{1,2t^2-1,2t^2(2t^2-1)}(x^{-1}q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q^2) \Bigg).\end{aligned} \tag{1.2.17}$$

Hence, $\mathcal{V}_t^{(m)}(x;q)$ is a mixed false theta series.

1.2.4 Modularity of $\mathcal{O}_t^{(m)}(x;q)$

For the fourth case, consider odd unimodal sequences, i.e., unimodal sequences where the parts a_i, b_j and c are odd positive integers. For example, there are six such sequences of weight 4, namely

$$(1,\overline{3}),\ (\overline{3},1),\ (\overline{1},1,1,1),\ (1,\overline{1},1,1),\ (1,1,\overline{1},1),\ (1,1,1,\overline{1}).$$

Again, the rank is $s - r$. Let $\mathrm{ou}(m,n)$ denote the number of odd unimodal sequences of weight n and rank m and consider its generating function [15, Eq. (1.5)]

$$\mathcal{O}(x;q) := \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} \mathrm{ou}(m,n) x^m q^n = \sum_{n\geq 0} \frac{q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \tag{1.2.18}$$

which satisfies the Hecke-Appell type expansion [15, Eq. (1.7)]

$$\mathcal{O}(x;q) = \frac{q}{(q^2;q^2)_\infty^2} \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) \frac{(-1)^{r+s} q^{r^2+4rs+s^2+3r+3s}}{1 - xq^{2r+1}}. \tag{1.2.19}$$

Remark 1.2.8. *One can confirm* (1.2.18) *using the ideas in Remark 1.2.2.*

For $t, m \in \mathbb{Z}$ with $t \geq 1, 1-t \leq m \leq t$, consider the generalisation

$$p_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^{r+s} q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)r+(t+m)s}}{1-xq^r} \tag{1.2.20}$$

and

$$\mathcal{O}_t^{(m)}(x;q) := \frac{q}{(q^2;q^2)_\infty^2} p_{t,m}(qx;q^2). \tag{1.2.21}$$

By (1.2.19)–(1.2.21), $\mathcal{O}_1^{(1)}(x;q) = \mathcal{O}(x;q)$. Our next result exhibits that $\mathcal{O}_t^{(m)}(x;q)$ is a mixed false theta function.

Theorem 1.2.9. *Let $t, m \in \mathbb{Z}$ with $t \geq 1, 1-t \leq m \leq t$. For generic x, we have*

$$\begin{aligned}\mathcal{O}_t^{(m)}(x;q) = \frac{q^{3-8t^2-2(m-1)(2t-1)}}{\Theta(xq;q^2)(q^2;q^2)_\infty^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{k^2+3k} f_{1,2t,1}(q^{2t+2m+2-8t^2+2k}, q^{2t+2m}; q^2) \\ \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{2k+1}, -q^{16t^4-4t^2-2(m-1)(2t-1)+8t^2k}; q^2).\end{aligned} \tag{1.2.22}$$

Hence, $\mathcal{O}_t^{(m)}(x;q)$ is a mixed false theta series.

1.2.5 Modularity of $V_t^{(m)}(x;q)$

For the final case, consider unimodal sequences where $\sum b_i$ is a partition into parts at most $c-k$ where k is the size of the Durfee square of the partition $\sum a_i$. For example, there are ten such sequences of weight 4, namely

$$(\overline{4}),\ (1,\overline{3}),\ (\overline{3},1),\ (1,\overline{2},1),\ (\overline{2},2),\ (2,\overline{2}),\ (1,1,\overline{2}),\ (\overline{2},1,1),\ (\overline{1},1,1,1),\ (1,1,1,\overline{1}).$$

Here, the rank is $s-r$ where the empty sequence has rank 0. Let $V(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [29, Eq. (3.1)]

$$V(x;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} V(m,n)x^m q^n = \sum_{n\geq 0} \frac{(q^{n+1})_n q^n}{(xq)_n (q/x)_n} \tag{1.2.23}$$

which satisfies [29, Eq. (3.3)]

$$V(x;q) = \frac{(1-x)}{(q)_\infty^2} \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^r q^{\binom{r}{2}+3rs+6\binom{s}{2}+2r+5s}(1-q^{r+2s+1})}{1-xq^r}. \tag{1.2.24}$$

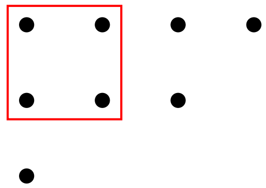

Figure 1.3: The partition $(4, 3, 1)$ with a Durfee square of size 2.

Remark 1.2.10. *A partition of n has a Durfee square of size s if s is the largest number such that the partition contains at least s parts with values $\geq$ s, e.g., see Figure 1.3. Kim and Lovejoy obtain* (1.2.23) *combinatorially [29, Prop. 3.1] by first altering the summand using the q-Chu-Vandermonde transformation, namely*

$$\frac{(q^{n+1})_n}{(qx, q/x)_n} = \underbrace{q^n}_{\bar{c}} \underbrace{\sum_{k=0}^{n} \frac{x^{-k} q^{k^2}}{(q/x)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q}_{\sum a_i} \underbrace{\frac{1}{(qx)_{n-k}}}_{\sum b_i}.$$

Then observe that any partition of size at most n can be defined by its Durfee square of size k, a partition of weight at most k and a partition contained within $k \times (n-k)$ rectangle. This is the $\sum a_i$ sequence to the left of the peak $\bar{c}$. The right-most sequence $\sum b_i$ is then expressed in terms of the Durfee square of size k. The generating function for $V(x; q)$ is therefore consistent with Figure 1.4.

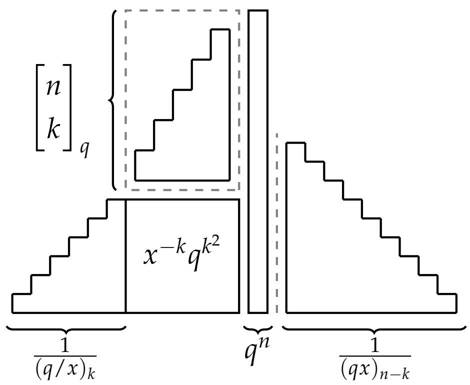

Figure 1.4: Construction of a sequence counted by $V(x; q)$.

For $t, m \in \mathbb{Z}$ with $t \geq 1$, $0 \leq m \leq 3t-1$, consider the generalisation

$$\ell_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^r q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s} \times \frac{1-q^{(3t-2m)r+2\binom{3t-1}{2}s+\binom{3t-1}{2}}}{1-xq^r} \tag{1.2.25}$$

and

$$V_t^{(m)}(x;q) := \frac{(1-x)}{(q)_\infty^2}\ell_{t,m}(x;q). \tag{1.2.26}$$

By (1.2.24)–(1.2.26), $V_1^{(1)}(x;q) = V(x;q)$. Before stating our last result, we recall the following triple sums [36]

$$\mathfrak{g}_{a,b,c,d,e,f}(x,y,z;q) := \left(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\right)(-1)^{r+s+t}x^r y^s z^t q^{a\binom{r}{2}+brs+c\binom{s}{2}+drt+est+f\binom{t}{2}} \tag{1.2.27}$$

where a, b, c, d, e and f are positive integers. These building blocks have appeared in the context of the modularity of coefficients of open Gromov-Witten potentials of elliptic orbifolds [16], unified WRT invariants of the Seifert manifolds constructed from rational surgeries on the left-handed torus knots $T^*_{(2,2t+1)}$ [27], false theta functions [30] and mock theta functions [52]. Our last result exhibits that $V_t^{(m)}(x;q)$ is a sum of mixed mock theta functions, mixed false theta series, the triple sums (1.2.27) and a modular form.

Theorem 1.2.11. *Let $t, m \in \mathbb{Z}$ with $t \geq 1$, $0 \leq m \leq 3t-1$. For generic x, we have*

$$
\begin{aligned}
V_t^{(m)}(x;q) &= \frac{(1-x)(-1)^t q^{(1-m)(1-3t)}}{(q)_\infty^2} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q) \\
&\quad \times \left(\frac{f_{1,1,3t}(x^{-1}q, (-1)^{t+1}q^{3tm-m+1}; q)}{\Theta(x;q)} - \frac{x^{-1} f_{1,1,3t}(xq, (-1)^{t+1}q^{3tm-m+1}; q)}{\Theta(x^{-1};q)} \right) \\
&\quad + \frac{(1-x)}{(q)_\infty^2} \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) \\
&\qquad \times \left(\frac{\mathfrak{g}_{1,1,3t,1,3t,1}(x^{-1}q, (-1)^{t+1}q^{3mt+1-m}, q^{i+1}; q)}{\Theta(x;q)} \right. \\
&\qquad\qquad \left. - \frac{x^{-1}\mathfrak{g}_{1,1,3t,1,3t,1}(xq, (-1)^{t+1}q^{3mt+1-m}, q^{i+1}; q)}{\Theta(x^{-1};q)} \right) \\
&\quad - \frac{\Theta(-q^{\binom{3t-1}{2}}; q^{3t(3t-1)})}{(q)_\infty^2}.
\end{aligned} \tag{1.2.28}
$$

Remark 1.2.12. *It is important to acknowledge that $V_t^{(m)}(x;q)$ can not be characterised as neatly as the four other functions. By examining the Hecke-type double sums, we identify both mixed false and mixed mock components. Also, it is not yet known if the triple sums in* (1.2.28) *have explicit evaluations in terms of Appell-Lerch sums or false theta series. However, instances of such triple sums have been identified as mixed false theta series [36].*

Remark 1.2.13. *The definitions of the two-parameter functions* (1.2.5)*,* (1.2.10)*,* (1.2.25)*,* (1.2.3) *and* (1.2.20) *may not be canonical but are constructed to satisfy two key properties. The first being that if*

$$
f_t^{(m)}(x;q) = \sum_{r,s\in\mathbb{Z}} sg(r,s)(-1)^{r+s} \frac{q^{Q(r,s)}}{1-xq^{L(r)}}
$$

is one of the functions mentioned above, $Q(r,s)$ is at most quadratic in r and s, and $L(r)$ is linear in r, then

$$
\sum_{r,s\in\mathbb{Z}} sg(r,s)(-1)^{r+s} q^{Q(r,s)}
$$

is a modular form for all eligible t and m. The second condition being that the recurrences $a_r - a_{r+d} = b_r + c_r$ seen in (3.1.15)*,* (3.2.12) *and* (3.3.12) *satisfy $c_r = 0$ for all $r \notin [-d,-1]$.*

For example, an alternative definition of (1.2.15)

$$k_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^{r+s} q^{2\binom{r}{2}+2trs+\binom{s}{2}+4tr+2ms}}{1-xq^{2r+1}} \tag{1.2.29}$$

implies $c_r \neq 0$ *when* $r \in [2t-1-2t^2, t-1]$ *where* $t \geq 1$, *i.e., the second condition fails.*

The rest of this thesis is organised as follows. In Chapter 2, we illustrate the role of Hecke-type double sums in characterising the modularity of q-series. In Chapter 3, we prove Theorems 1.2.3, 1.2.5, 1.2.7, 1.2.9 and 1.2.11. In Chapter 4, we demonstrate that the base cases of these results recover the partial theta identities (1.0.6)–(1.0.9). Finally, in Chapter 5, we discuss some future directions of this work.

Chapter 2

Preliminaries

In order to prove the mixed mock (false) modularity of a given q-series, one can explicitly express it as a linear combination of terms of the form $h_i g_i$ where h_i is a modular form, and g_i is a known mixed mock modular (mixed false theta) building block. The Hecke-type double sum (1.1.5) is a universal building block of which the modularity can be determined.

2.1 Characterisation of Modularity

The results in this thesis utilise the fact that we can express the generating functions in (1.2.3), (1.2.5), (1.2.7), (1.2.9) and (1.2.11) in terms of Hecke-type double sums (1.1.5). Note that a, b, c may be positive rational numbers in (1.1.5) however we can make the parameters integers by observing

$$f_{a,b,c}(x, y; q) = f_{\lambda a, \lambda b, \lambda c}(x; y; q^{\frac{1}{\lambda}})$$

where λ is the least common multiple of the denominators of a, b and c.

Sums of this form are useful because we can determine their modularity. Define the *discriminant* of $f_{a,b,c}$ as $D := b^2 - ac$. In 2014, Hickerson and Mortenson studied specialisations of $a, b, c \in \mathbb{Z}_{>0}$ with $D > 0$ to reprove the mock theta conjectures. In 2023, Mortenson and Zwegers proved the following formula for $D > 0$ for arbitrary positive integers a, b, c.

Theorem 2.1.1 ([35], Corollary 4.2). *For $D := b^2 - ac > 0$ and generic x and y, we have*

$$f_{a,b,c}(x,y;q) = g_{a,b,c}(x,y,-1,-1;q) + \frac{1}{\Theta(-1;q^{aD})\Theta(-1;q^{cD})}\vartheta_{a,b,c}(x,y;q) \tag{2.1.1}$$

where

$$\begin{aligned} g_{a,b,c}(x,y,z_1,z_0;q) := &\sum_{i=0}^{a-1}(-y)^i q^{c\binom{i}{2}}\Theta(q^{bi}x;q^a)m\left(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-iD}\frac{(-y)^a}{(-x)^b}, z_0; q^{aD}\right) \\ &+ \sum_{i=0}^{c-1}(-x)^i q^{a\binom{i}{2}}\Theta(q^{bi}y;q^c)m\left(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-iD}\frac{(-x)^c}{(-y)^b}, z_1; q^{cD}\right), \end{aligned}$$

$$\begin{aligned} \vartheta_{a,b,c}(x,y;q) := &\sum_{d^*=0}^{b-1}\sum_{e^*=0}^{b-1} q^{a\binom{d-c/2}{2}+b(d-c/2)(e+a/2)+c\binom{e+a/2}{2}}(-x)^{d-c/2}(-y)^{e+a/2} \\ &\times \sum_{f=0}^{b-1} q^{ab^2\binom{f}{2}+\left(a(bd+b^2+ce)-ac(b+1)/2\right)f}(-y)^{af}\Theta(-q^{c(ad+be+a(b-1)/2+abf)}(-x)^c;q^{cb^2}) \\ &\times \Theta(-q^{a((d+b(b+1)/2+bf)D+c(a-b)/2)}(-x)^{-ac}(-y)^{ab};q^{ab^2D}) \\ &\times \frac{(q^{bD};q^{bD})_\infty^3\Theta(q^{D(d+e)+ac-b(a+c)/2}(-x)^{b-c}(-y)^{b-a};q^{bD})}{\Theta(q^{De+a(c-b)/2}(-x)^b(-y)^{-a};q^{bD})\,\Theta(q^{Dd+c(a-b)/2}(-y)^b(-x)^{-c};q^{bD})}, \end{aligned}$$

$d := d^* + \{c/2\}$ *and* $e := e^* + \{a/2\}$ *with* $0 \le \{\alpha\} < 1$ *denoting the fractional part of* α.

Theorem 2.1.2 ([37], Theorem 1.4). *For $D := b^2 - ac < 0$, we have*

$$\begin{aligned} f_{a,b,c}(x,y;q) = \frac{1}{2}\Bigg(&\sum_{i=0}^{a-1}(-y)^i q^{c\binom{i}{2}}\Theta(q^{bi}x;q^a)\sum_{r\in\mathbb{Z}} sg(r)\left(q^{a\binom{b+1}{2}-c\binom{a+1}{2}-iD}\frac{(-y)^a}{(-x)^b}\right)^r q^{-aD\binom{r+1}{2}} \\ &+ \sum_{i=0}^{c-1}(-x)^i q^{a\binom{i}{2}}\Theta(q^{bi}y;q^c)\sum_{r\in\mathbb{Z}} sg(r)\left(q^{c\binom{b+1}{2}-a\binom{c+1}{2}-iD}\frac{(-x)^c}{(-y)^b}\right)^r q^{-cD\binom{r+1}{2}}\Bigg). \end{aligned} \tag{2.1.2}$$

Given Theorems 2.1.1 and 2.1.2, it is natural to ask "What modularity-type occurs when $D = 0$?". In this case, we prove that $f_{a,b,c}$ is a modular form. We prove this by first showing that $f_{1,1,1}(x,y;q)$ is a modular form. We then show that $f_{a,b,c}(x,y;q)$ can be expressed in terms of $f_{1,1,1}$'s for all $a,b,c \in \mathbb{Z}_{>0}$ such that $D = 0$.

Lemma 2.1.3. *Let $x, y \in \mathbb{C}^\times$ be generic. Then*

$$f_{1,1,1}(x,y;q) = \frac{\Theta(y;q)}{1-\frac{x}{y}} + \frac{\Theta(x;q)}{1-\frac{y}{x}}. \tag{2.1.3}$$

Proof. Using (1.1.5), we have

$$\begin{aligned} f_{1,1,1}(x,y;q) &= \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} q^{\binom{r}{2}+rs+\binom{s}{2}} x^r y^s \\ &= \sum_{r,s\geq 0} (-1)^{r+s} q^{\binom{r+s}{2}} x^r y^s - \sum_{r,s<0} (-1)^{r+s} q^{\binom{r+s}{2}} x^r y^s \\ &\qquad + \sum_{r\geq 0, s<0} (-1)^{r+s} q^{\binom{r+s}{2}} x^r y^s - \sum_{r\geq 0, s<0} (-1)^{r+s} q^{\binom{r+s}{2}} x^r y^s \\ &= \sum_{r\geq 0} x^r \sum_{s\in\mathbb{Z}} (-1)^{r+s} q^{\binom{r+s}{2}} y^s - \sum_{s<0} y^s \sum_{r\in\mathbb{Z}} (-1)^{r+s} q^{\binom{r+s}{2}} x^r. \end{aligned} \tag{2.1.4}$$

Taking $s \mapsto s - r$ in the first sum in (2.1.4) and $r \mapsto r - s$ in the second sum in (2.1.4), we obtain

$$\begin{aligned} &\sum_{r\geq 0} x^r \sum_{s} (-1)^s q^{\binom{s}{2}} y^{s-r} - \sum_{s<0} y^s \sum_{r} (-1)^r q^{\binom{r}{2}} x^{r-s} \\ &\qquad = \sum_{r\geq 0} \left(\frac{x}{y}\right)^r \Theta(y;q) - \sum_{s<0} \left(\frac{y}{x}\right)^s \Theta(x;q) \\ &\qquad = \sum_{r\geq 0} \left(\frac{x}{y}\right)^r \Theta(y;q) - \sum_{s\geq 0} \left(\frac{y}{x}\right)^{-s-1} \Theta(x;q) \\ &\qquad = \frac{\Theta(y;q)}{1-\frac{x}{y}} - \frac{\left(\frac{x}{y}\right)\Theta(x;q)}{1-\frac{x}{y}} \\ &\qquad = \frac{\Theta(y;q)}{1-\frac{x}{y}} + \frac{\Theta(x;q)}{1-\frac{y}{x}}. \end{aligned}$$

□

Theorem 2.1.4. *Let $a, b \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Q}_{>0}$ such that $D := b^2 - ac = 0$ and $gcd(a,b) = 1$. Also, let*

$$\bar{x} := \begin{cases} \zeta_{2a} x^{\frac{1}{a}} q^{-\frac{1-a}{2a}} & \text{if } a \text{ even}, \\ x^{\frac{1}{a}} q^{-\frac{1-a}{2a}} & \text{if } a \text{ odd} \end{cases}; \qquad \bar{y} := \begin{cases} \zeta_{2b} y^{\frac{1}{b}} q^{-\frac{1-b}{2a}} & \text{if } b \text{ even}, \\ y^{\frac{1}{b}} q^{-\frac{1-b}{2a}} & \text{if } b \text{ odd} \end{cases}$$

where $\zeta_n := e^{\frac{2\pi i}{n}}$ and l_n, k_n are Bézout coefficients which arise as a solution of $al_n + bk_n = n$. Then

$$f_{a,b,c}(x,y;q) = \left(\sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \left(\frac{\bar{x}}{\bar{y}} \right)^{bk_{i+j}-j} \right)^{-1} \Bigg\{ f_{1,1,1}(\bar{x}, \bar{y}; q^{\frac{1}{a}}) + \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \left(\frac{\bar{x}}{\bar{y}} \right)^{bk_{i+j}-j} \tag{2.1.5}$$

$$\times \left[\sum_{m=0}^{l_{i+j}-1} (-x)^m q^{a\binom{m}{2}} \Theta(yq^{bm}; q^c) + \sum_{m=0}^{k_{i+j}-1} (-y)^m q^{c\binom{m}{2}} \Theta(xq^{bm}; q^a) \right] \Bigg\}. \tag{2.1.6}$$

In order to prove Theorem 2.1.4, we make use of the following result ([25], Prop. 5.3).

Proposition 2.1.5. *For $x, y \in \mathbb{C}^*$ and $l, k \in \mathbb{Z}$*

$$f_{a,b,c}(x,y,q) = (-x)^l (-y)^k q^{a\binom{l}{2} + blk + c\binom{k}{2}} f_{a,b,c}(q^{al+bk}x, q^{bl+ck}y, q)$$
$$+ \sum_{m=0}^{l-1} (-x)^m q^{a\binom{m}{2}} \Theta(q^{mb}y; q^c) + \sum_{m=0}^{k-1} (-y)^m q^{c\binom{m}{2}} \Theta(q^{mb}x; q^a), \tag{2.1.7}$$

Proof of Theorem 2.1.4. Firstly, since $D = 0$ and by scaling the indices of $f_{a,b,c}$, we are reduced to studying

$$f_{a,b,c}(x,y;q) = f_{a,b,\frac{b^2}{a}}(x,y;q) \tag{2.1.8}$$
$$= f_{a^2,ab,b^2}(x,y;q^{\frac{1}{a}}). \tag{2.1.9}$$

We show that we can express f_{a^2,ab,b^2} in terms of $f_{1,1,1}$ and then compute $f_{1,1,1}$ explicitly.

Taking $(r,s) \mapsto (ar+i, bs+j)$ for $0 \leq i \leq a-1$ and $0 \leq j \leq b-1$ yields

$$\begin{aligned}
f_{1,1,1}(x,y;q) &= \sum_{r,s} \mathrm{sg}(r,s)(-1)^{r+s} q^{\binom{r}{2}+rs+\binom{s}{2}} x^r y^s \\
&= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{r,s} \mathrm{sg}(ar+i,bs+j)(-1)^{ar+i+bs+j} q^{\binom{ar+i}{2}+(ar+i)(bs+j)+\binom{bs+j}{2}} x^{ar+i} y^{bs+j} \\
&= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{r,s} \mathrm{sg}(r,s)(-1)^{ar+i+bs+j} \\
&\qquad \times q^{a^2\binom{r}{2}+abrs+b^2\binom{s}{2}+r\left(\binom{a}{2}+ai+aj\right)+s\left(\binom{b}{2}+bi+bj\right)+\binom{i}{2}+\binom{j}{2}+ij} x^{ar+i} y^{bs+j} \\
&= \sum_{i=0}^{a-1}(-1)^i x^i q^{\binom{i}{2}} \sum_{j=0}^{b-1}(-1)^j y^j q^{\binom{j}{2}+ij} \sum_{r,s}\mathrm{sg}(r,s)(-1)^{r+s} \\
&\qquad \times q^{a^2\binom{r}{2}+abrs+b^2\binom{s}{2}+r\left(\binom{a}{2}+a(i+j)\right)+s\left(\binom{b}{2}+b(i+j)\right)} \left((-1)^{a-1}x^a\right)^r \left((-1)^{b-1}y^b\right)^s \\
&= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{i+j}x^i y^j q^{\binom{i+j}{2}} f_{a^2,ab,b^2}\left((-1)^{a-1}x^a q^{\binom{a}{2}+a(i+j)}, (-1)^{b-1}y^b q^{\binom{b}{2}+b(i+j)};q\right).
\end{aligned} \tag{2.1.10}$$

The third equality in (2.1.10) uses the fact that $\mathrm{sg}(ar+i,s) = \mathrm{sg}(r,s) = \mathrm{sg}(r,bs+j)$ when $0 \leq i \leq a-1$ and $0 \leq j \leq b-1$. Let $l = l_{i+j}, k = k_{i+j}$ as in Proposition 2.1.5 so that $al_{i+j} + bk_{i+j} = i+j$. Then by Proposition (2.1.5),

$$\begin{aligned}
&f_{a^2,ab,b^2}\left((-1)^{a-1}x^a q^{\binom{a}{2}}, (-1)^{b-1}y^b q^{\binom{b}{2}};q\right) \\
&= \left((-1)^a x^a q^{\binom{a}{2}}\right)^{l_{i+j}} \left((-1)^b y^b q^{\binom{b}{2}}\right)^{k_{i+j}} q^{a^2\binom{l_{i+j}}{2}+abl_{i+j}k_{i+k}+b^2\binom{k_{i+j}}{2}} \\
&\qquad \times f_{a^2,ab,b^2}\left((-1)^{a-1}x^a q^{\binom{a}{2}+a(i+j)}, (-1)^{b-1}y^b q^{\binom{b}{2}+b(i+j)};q\right) \\
&+ \sum_{m=0}^{l_{i+j}-1}(-1)^{am}x^{am}q^{a^2\binom{m}{2}+m\binom{a}{2}}\Theta\left((-1)^{b-1}y^b q^{\binom{b}{2}+abm};q^{b^2}\right) \\
&\qquad + \sum_{m=0}^{k_{i+j}-1}(-1)^{bm}y^{bm}q^{b^2\binom{m}{2}+m\binom{b}{2}}\Theta\left((-1)^{a-1}x^a q^{\binom{a}{2}+abm};q^{a^2}\right).
\end{aligned} \tag{2.1.11}$$

Rearranging yields

$$
\begin{aligned}
&f_{a^2,ab,b^2}((-1)^{a-1}x^a q^{\binom{a}{2}+a(i+j)}, (-1)^{b-1}y^b q^{\binom{b}{2}+b(i+j)}; q) \\
&\quad = (-1)^{al_{i+j}+bk_{i+j}} x^{-al_{i+j}} y^{-bk_{i+j}} q^{-a^2\binom{l_{i+j}}{2}-abl_{i+j}k_{i+k}-b^2\binom{k_{i+j}}{2}-l_{i+j}\binom{a}{2}-k_{i+j}\binom{b}{2}} \\
&\qquad \times \Big\{ f_{a^2,ab,b^2}((-1)^{a-1}x^a q^{\binom{a}{2}}, (-1)^{b-1}y^b q^{\binom{b}{2}}; q) \\
&\qquad - \sum_{m=0}^{l_{i+j}-1} (-1)^{am} x^{am} q^{a^2\binom{m}{2}+m\binom{a}{2}} \Theta\left((-1)^{b-1}y^b q^{\binom{b}{2}+abm}; q^{b^2}\right) \\
&\qquad - \sum_{m=0}^{k_{i+j}-1} (-1)^{bm} y^{bm} q^{b^2\binom{m}{2}+m\binom{b}{2}} \Theta\left((-1)^{a-1}x^a q^{\binom{a}{2}+abm}; q^{a^2}\right) \Big\}.
\end{aligned}
\tag{2.1.12}
$$

We can apply $al_{i+j} + bk_{i+j} = i + j$ to simplify the first term on the right-hand side to

$$
(-1)^{i+j} \left(\frac{x}{y}\right)^{bk_{i+j}} q^{-\binom{i+j}{2}} \tag{2.1.13}
$$

and substitute (2.1.12) into (2.1.10) to obtain

$$
\begin{aligned}
&f_{1,1,1}(x,y;q) = f_{a^2,ab,b^2}((-1)^{a-1}x^a q^{\binom{a}{2}}, (-1)^{b-1}y^b q^{\binom{b}{2}}; q) \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} \left(\frac{x}{y}\right)^{bk_{i+j}-j} \\
&- \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} \left(\frac{x}{y}\right)^{bk_{i+j}-j} \left\{ \sum_{m=0}^{l_{i+j}-1} (-1)^{am} x^{am} q^{a^2\binom{m}{2}+m\binom{a}{2}} \Theta\left((-1)^{b-1}y^b q^{\binom{b}{2}+abm}; q^{b^2}\right) \right. \\
&\left. + \sum_{m=0}^{k_{i+j}-1} (-1)^{bm} y^{bm} q^{b^2\binom{m}{2}+m\binom{b}{2}} \Theta\left((-1)^{a-1}x^a q^{\binom{a}{2}+abm}; q^{a^2}\right) \right\}.
\end{aligned}
\tag{2.1.14}
$$

So, we can express f_{a^2,ab,b^2} in terms of $f_{1,1,1}$ as follows:

$$
\begin{aligned}
&f_{a^2,ab,b^2}((-1)^{a-1}x^a q^{\binom{a}{2}}, (-1)^{b-1}y^b q^{\binom{b}{2}}; q) = \left(\sum_{j=0}^{b-1} \left(\frac{x}{y}\right)^{bk_{i+j}-j} \right)^{-1} \Big\{ f_{1,1,1}(x,y;q) \\
&\quad + \sum_{j=0}^{b-1} \left(\frac{x}{y}\right)^{bk_{i+j}-j} \left[\sum_{m=0}^{l_{i+j}-1} (-1)^{am} x^{am} q^{a^2\binom{m}{2}+m\binom{a}{2}} \Theta\left((-1)^{b-1}y^b q^{\binom{b}{2}+abm}; q^{b^2}\right) \right. \\
&\quad \left. + \sum_{m=0}^{k_{i+j}-1} (-1)^{bm} y^{bm} q^{b^2\binom{m}{2}+m\binom{b}{2}} \Theta\left((-1)^{a-1}x^a q^{\binom{a}{2}+abm}; q^{a^2}\right) \right] \Big\}.
\end{aligned}
\tag{2.1.15}
$$

Combining (2.1.3) and (2.1.8), and taking $(x,y) \mapsto (\bar{x},\bar{y})$ in (2.1.15) yields the result. □

Remark 2.1.6. *When $gcd(a,b) = \lambda \neq 1$, we can still use Theorem 2.1.4 by observing that*

$$f_{a,b,c}(x,y;q) = f_{\frac{a}{\lambda},\frac{b}{\lambda},\frac{c}{\lambda}}(x,y;q^{\lambda}).$$

For instance, $f_{4,6,9}(x,y;q) = f_{2,3,\frac{9}{2}}(x,y;q^2)$.

Remark 2.1.7. *In summary, Theorems 2.1.1, 2.1.2 and 2.1.4 state:*

- *If $D > 0$ then $f_{a,b,c}$ is mixed mock;*
- *If $D < 0$ then $f_{a,b,c}$ is mixed false; and*
- *If $D = 0$ then $f_{a,b,c}$ is a modular form.*

We next address a result concerning identities satisfied by $\mathrm{sg}(r,s)$. We omit the proof. Let

$$\delta(r) := \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1.8. *For r, s, $t \in \mathbb{Z}$ with $t \geq 1$, we have*

$$\mathrm{sg}(-r,-s-1) = -\mathrm{sg}(r,s) + \delta(r) \tag{2.1.16}$$

and

$$\mathrm{sg}(r-1,s+2t) = \mathrm{sg}(r,s) - \delta(r) + \sum_{i=1}^{2t} \delta(s+i). \tag{2.1.17}$$

$$\mathrm{sg}(r-(3t-1),s+1) = \mathrm{sg}(r,s) - \sum_{i=0}^{3t-2} \delta(r-i) + \delta(s+1) \tag{2.1.18}$$

and

$$\mathrm{sg}(r,l)\,\mathrm{sg}(r+3tl,s) = \mathrm{sg}(r,l)\,\mathrm{sg}(r,s). \tag{2.1.19}$$

We now recall the theta function identities

$$\Theta(q^n;q) = 0, \tag{2.1.20}$$

$$\Theta(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} \Theta(x;q) \tag{2.1.21}$$

and

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{\frac{1}{2}k^2+(n+\frac{1}{2})k}}{1-xq^k} = \frac{(q)_\infty^3}{x^n \Theta(x;q)} \tag{2.1.22}$$

where $n \in \mathbb{Z}$. While (2.1.20) and (2.1.21) easily follow from the definition of the theta function, (2.1.22) follows from combining [7, Entry 3.2.1] with (2.1.20). Next, we turn to providing alternative expressions for (1.2.10) and (1.2.15) which will be beneficial in the proofs of Theorems 1.2.5 and 1.2.7, respectively. Let

$$\mathcal{H}_t^{(m)}(x;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}}{1-xq^r}, \tag{2.1.23}$$

$$\kappa_{t,m}(x;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}}{1-xq^{2r+1}} \tag{2.1.24}$$

and

$$\Phi_t^{(m)}(x;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r}\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}}{1-xq^r}. \tag{2.1.25}$$

Proposition 2.1.9. *We have*

$$h_{t,m}(x;q) = \mathcal{H}_t^{(m)}(x;q) - x^{-1}\mathcal{H}_t^{(m)}(x^{-1};q), \tag{2.1.26}$$

$$k_{t,m}(x;q) = x^{-1}q^{-1}\left(\kappa_{t,m}(x;q) - f_{2,2t,1}(q^{2t},q^{2m};q)\right). \tag{2.1.27}$$

and

$$\ell_{t,m}(x;q) = \Phi_t^{(m)}(x;q) - x^{-1}\Phi_t^{(m)}(x^{-1};q) - \frac{1}{1-x}\Theta(-q^{\binom{3t-1}{2}};q^{3t(3t-1)}). \tag{2.1.28}$$

Proof. We first let $(r,s)\to(-r-1,-s-1)$ in $\mathcal{H}_t^{(m)}(x^{-1};q)$ and simplify to obtain

$$-xq^{1+m+t}\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(-r-1,-s-1)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(m+t+1)r+2ts}}{1-xq^{r+1}}. \tag{2.1.29}$$

Next, applying $r\to r-1$ to (2.1.29), then using (1.0.1) and (2.1.16) yields

$$-x\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(m+t)r}}{1-xq^r}$$

and thus (2.1.26) follows. Using (1.1.5), one observes that

$$f_{2,2t,1}(q^{2t},q^{2m};q) = \kappa_{t,m}(x;q) - xqk_{t,m}(x;q)$$

and so (2.1.27) follows. Now, let $(r,s) \to (-r-1,-s-1)$ in $\Phi_t^{(m)}(x^{-1};q)$ and simplify to get

$$xq \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(-r-1,-s-1)(-1)^r \times \frac{q^{\binom{r+2}{2}+3t(r+1)(s+1)+3t(3t-1)\binom{s+2}{2}-(m+1)(r+1)-(\binom{3t}{2}+3t-1)(s+1)+r}}{1-xq^{r+1}}. \tag{2.1.30}$$

Finally, applying $r \to r-1$ to (2.1.30), then (1.0.1), (2.1.16) and (2.1.21) yields

$$x \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^r \frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(3t-m+1)r+\frac{(3t-1)(9t-2)}{2}s+\binom{3t-1}{2}}}{1-xq^r} + \frac{x}{1-x}\Theta(-q^{\binom{3t-1}{2}};q^{3t(3t-1)})$$

and so (2.1.28) follows. □

Chapter 3

Proofs of Main Results

The method of proof is as follows [35]. First, we derive functional equations for each of (1.2.5), (1.2.10), (1.2.15), (1.2.20), (1.2.25),

$$\hat{g}_{t,m}(x;q) := \Theta(x;q)g_{t,m}(x;q), \tag{3.0.1}$$

$$\hat{\mathcal{H}}_t^{(m)}(x;q) := \Theta(x;q)\mathcal{H}_t^{(m)}(x;q), \tag{3.0.2}$$

$$\hat{\kappa}_{t,m}(x;q) := \Theta(qx;q^2)\kappa_{t,m}(x;q), \tag{3.0.3}$$

$$\hat{p}_{t,m}(x;q) := \Theta(x;q)p_{t,m}(x;q) \tag{3.0.4}$$

and

$$\hat{\Phi}_t^{(m)}(x;q) := \Theta(x;q)\Phi_t^{(m)}(x;q). \tag{3.0.5}$$

Suitable care is required in constructing the sums (3.1.3), (3.2.3), (3.3.3), (3.4.3) and (3.5.3) which favorably decompose in order to obtain these functional equations. We then express each of (3.0.1)–(3.0.5) as a Laurent series in $x \in \mathbb{C} \setminus \{0\}$ and use the functional equations to find an explicit formula for the coefficients in the Laurent series expansion. After some sitzfleisch, these calculations eventually yield the right-hand sides of (1.2.7), (1.2.12), (1.2.17), (1.2.22) and (1.2.28).

3.1 Proof of Theorem 1.2.3 $\left(\text{Modularity of } \mathcal{U}_t^{(m)}(x;q)\right)$

For the first case, we begin with the following result.

Proposition 3.1.1. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $t \equiv m \pmod 2$, we have*

$$\begin{aligned} g_{t,m}(qx;q) &= -x^{1-4t^2} q^{\frac{t-m-2t^2-2tm}{2}} g_{t,m}(x;q) \\ &\quad - x^{1-4t^2-\frac{t+m}{2}} q^{\frac{t-m-2t^2-2tm}{2}} \frac{(q)_\infty^3}{\Theta(x;q)} \sum_{i=1}^{2t} (-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i} x^{2ti} \\ &\quad - x^{1-4t^2} q^{1-4t^2} \sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \end{aligned} \tag{3.1.1}$$

and

$$\begin{aligned} \hat{g}_{t,m}(qx;q) &= x^{-4t^2} q^{\frac{t-m-2t^2-2tm}{2}} \hat{g}_{t,m}(x;q) \\ &\quad + x^{-4t^2-\frac{t+m}{2}} q^{\frac{t-m-2t^2-2tm}{2}} (q)_\infty^3 \sum_{i=1}^{2t} (-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i} x^{2ti} \\ &\quad + x^{-4t^2} q^{1-4t^2} \Theta(x;q) \sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q). \end{aligned} \tag{3.1.2}$$

Proof. The idea is to compute the sum

$$x^{4t^2-1} q^{\frac{2t^2-t+2tm+m}{2}} \sum_{r,s\in\mathbb{Z}} \operatorname{sg}(r,s)(-1)^{r+s} q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s} \frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}} \tag{3.1.3}$$

in two ways. Expanding the numerator yields

$$x^{4t^2-1} q^{\frac{2t^2-t+2tm+m}{2}} g_{t,m}(qx;q) - \sum_{r,s\in\mathbb{Z}} \operatorname{sg}(r,s)(-1)^{r+s} \frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s+1-3t^2-\frac{t}{2}+tm+\frac{m}{2}}}{1-xq^{r+1}}. \tag{3.1.4}$$

Taking $(r,s) \to (r-1, s+2t)$ in the second sum in (3.1.4) and using (2.1.17), (2.1.20) and (2.1.22) leads to

$$\begin{aligned}
&-\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s}}{1-xq^r}\\
&=-\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s}}{1-xq^r}+\frac{1}{1-x}\sum_{s\in\mathbb{Z}}(-1)^s q^{\frac{s^2+(1+t-m)s}{2}}\\
&-\sum_{i=1}^{2t}\sum_{r\in\mathbb{Z}}(-1)^{r-i}\frac{q^{\frac{r^2}{2}-2tri+\frac{i^2}{2}+\frac{1+t+m}{2}r-\frac{1+t-m}{2}i}}{1-xq^r}\\
&=-g_{t,m}(x;q)-\frac{x^{-\frac{t+m}{2}}(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i}x^{2ti}.
\end{aligned} \tag{3.1.5}$$

Alternatively, we use

$$\frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}}=-x^{1-4t^2}q^{(r+1)(1-4t^2)}\sum_{k=0}^{4t^2-2}x^k q^{k(r+1)} \tag{3.1.6}$$

to express (3.1.3) as

$$\begin{aligned}
&-q^{\frac{-6t^2-t+2tm+m}{2}+1}\sum_{k=0}^{4t^2-2}x^k q^k\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m+2-8t^2+2k}{2}r+\frac{1+t-m}{2}r+r(1-4t^2)}\\
&=-q^{\frac{-6t^2-t+2tm+m}{2}+1}\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q).
\end{aligned} \tag{3.1.7}$$

Combining (3.1.4), (3.1.5) and (3.1.7) gives us (3.1.1). Finally, (3.1.2) follows from (2.1.21), (3.0.1) and (3.1.1). □

We are now in a position to prove our first result.

Proof of Theorem 1.2.3. Note that $\hat{g}_{t,m}(x)=\hat{g}_{t,m}(x;q)$ does not have poles and so we may write

$$\hat{g}_{t,m}(x)=\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r}a_r x^{-r} \tag{3.1.8}$$

for all $x \in \mathbb{C} \setminus \{0\}$. Substituting (3.1.8) into (3.1.2), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r-r} a_r x^{-r} &= x^{-4t^2} q^{\frac{t-m-2t^2-2tm}{2}} \sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} a_r x^{-r} \\ &+ x^{-4t^2-\frac{t+m}{2}} q^{\frac{t-m-2t^2-2tm}{2}} (q)_\infty^3 \sum_{i=1}^{2t}(-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i} x^{2ti} \\ &+ x^{-4t^2} q^{1-4t^2}\Theta(x;q) \\ &\quad\times \sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}};q).\end{aligned} \tag{3.1.9}$$

Using

$$\binom{a-b}{2} = \binom{a}{2} - ab + \binom{b+1}{2} \tag{3.1.10}$$

and (1.0.1), one can check that the last sum on the right-hand side of (3.1.9) can be written as

$$q^{1-4t^2}\sum_{r\in\mathbb{Z}}(-1)^r q^{\binom{r+1}{2}} \sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+r(k-4t^2)+k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}};q) x^{-r}. \tag{3.1.11}$$

We now let $r \to r-4t^2$ in the first term on the right-hand side of (3.1.9), apply (3.1.11) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b'_r + c'_r \tag{3.1.12}$$

where

$$\begin{aligned} b'_r := q^{1-4t^2+\binom{r+1}{2}-\frac{r^2}{8t^2}-\frac{t-m+2t^2-2tm-8t^2}{8t^2}r-4t^2 r} &\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+k(r+1)} \\ &\times f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}};q)\end{aligned}$$

and

$$c'_r := (-1)^{i+\frac{t+m}{2}}(q)_\infty^3 q^{\frac{i^2}{2}-\frac{1+t-m}{2}i-\frac{\left(4t^2+\frac{t+m}{2}-2ti\right)^2}{8t^2}-\frac{t-m+2t^2-2tm-8t^2}{8t^2}\left(4t^2+\frac{t+m}{2}-2ti\right)+\frac{t-m-2t^2-2tm}{2}}$$

if $r = 4t^2+\frac{t+m}{2}-2ti$, $1 \le i \le 2t$, and is 0 otherwise. Moreover, using (1.2.5), (3.0.1) and Cauchy's integral formula applied to (3.1.8), a short calculation gives

$$a_r = -\frac{1}{2\pi i} q^{\frac{4t^2-1}{8t^2}r^2 - \frac{t-m+2t^2-2tm}{8t^2}r} \oint \sum_{\lambda\in\mathbb{Z}} (-1)^\lambda q^{\binom{\lambda+1}{2}-\lambda r} \times \sum_{n,s\in\mathbb{Z}} \mathrm{sg}(n,s)(-1)^{n+s} \frac{q^{\frac{n^2}{2}+2tns+\frac{s^2}{2}+\frac{t+1+m}{2}n+\frac{t+1-m}{2}s}}{1-zq^n}\, dz \tag{3.1.13}$$

where the integration is over a closed contour around 0 in $\mathbb{C}$. Thus, as $|q| < 1$,

$$\lim_{r\to\pm\infty} a_r = 0. \tag{3.1.14}$$

Now, observe that (3.1.12) is equivalent to

$$a_r - a_{r+4t^2} = b_r + c_r \tag{3.1.15}$$

where $b_r := -b'_{r+4t^2}$ and $c_r := -c'_{r+4t^2}$. We now claim that

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}. \tag{3.1.16}$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} a_r$ and use (3.1.15) to obtain

$$\alpha_r = q^{-r-2t^2-\frac{t-m+2t^2-2tm}{8t^2}} \alpha_{r+4t^2} + q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} b_r + q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} c_r. \tag{3.1.17}$$

In fact, we will demonstrate

$$\alpha_r = q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

which clearly implies (3.1.16). Let

$$\tilde{a}_r := \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.1.15) and (3.1.17), respectively. The former follows from

$$\begin{aligned} \tilde{a}_r - \tilde{a}_{r+4t^2} &= \sum_{l\in\mathbb{Z}} \left(\mathrm{sg}(r,l) - \mathrm{sg}(r+4t^2, l-1)\right) b_{r+4t^2 l} \\ &= \sum_{l\in\mathbb{Z}} \left(\delta(l) - \delta(r+1) - \cdots - \delta(r+4t^2)\right) b_{r+4t^2 l} \\ &= b_r - \left(\delta(r+1) + \cdots + \delta(r+4t^2)\right) \sum_{n\equiv r \pmod{4t^2}} b_n \end{aligned} \tag{3.1.18}$$

and

$$\sum_{n\equiv r \pmod{4t^2}} b_n = \sum_{n\equiv r \pmod{4t^2}} (a_n - a_{n+4t^2} - c_n) = - \sum_{n\equiv r \pmod{4t^2}} c_n = -c_r \tag{3.1.19}$$

where we have used (3.1.14), the definitions of c_r and c_r' and that $-t \le m \le 3t-2$. Now, since $\lim_{r\to\pm\infty} \alpha_r = 0$ and $\lim_{r\to\infty} \tilde{\alpha}_r = 0$, we have

$$\lim_{r\to\infty} (\alpha_r - \tilde{\alpha}_r) = 0. \tag{3.1.20}$$

Finally, we compute

$$\alpha_r - \tilde{\alpha}_r - q^{r+2t^2+\frac{t-m+2t^2-2tm}{8t^2}} \left(\alpha_{r-4t^2} - \tilde{\alpha}_{r-4t^2}\right) = 0$$

which in combination with (3.1.20) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$\begin{aligned}
\hat{g}_{t,m}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2l} x^{-r} \\
&= -q^{1-4t^2} \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r+4t^2(l+1)+1}{2} - \frac{(r+4t^2(l+1))^2}{8t^2} - \frac{t-m+2t^2-2tm-8t^2}{8t^2}(r+4t^2(l+1))} \\
&\quad \times q^{-4t^2(r+4t^2(l+1))} \sum_{k=0}^{4t^2-2} (-1)^k q^{\binom{k+1-4t^2}{2}+k(r+4t^2(l+1)+1)+\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \\
&\qquad\qquad \times f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) x^{-r} \\
&= q^{1-t^2-8t^4-\frac{t-m}{2}+tm} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1-4t^2}{2}+k+4t^2k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \\
&\quad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\frac{r^2}{2}+(4t^2-1)rl+2t^2l^2(4t^2-1)+(k+\frac{1}{2})r+(t^2-\frac{t-m}{2}+tm+4t^2k)l} x^{-r} \\
&= q^{1-3t^2-\frac{t-m}{2}+tm} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \\
&\qquad \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{4t^2k-t^2+tm-\frac{t-m}{2}+8t^4}; q).
\end{aligned} \tag{3.1.21}$$

Thus, (1.2.7) follows from (1.2.6), (3.0.1) and (3.1.21). We now apply (2.1.1) and (2.1.20) to deduce that $f_{1,2t,1}$ is a modular form and (2.1.2) to obtain that $f_{1,4t^2-1,4t^2(4t^2-1)}$ is a false theta function. □

3.2 Proof of Theorem 1.2.5 $\left(\textbf{Modularity of } W_t^{(m)}(x;q)\right)$

For the second case, we begin with the following result.

Proposition 3.2.1. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned}\mathcal{H}_t^{(m)}(qx;q) &= -x^{1-4t^2}q^{m-2t^2}\mathcal{H}_t^{(m)}(x;q) - x^{1-4t^2+m-t}q^{m-2t^2}\frac{(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^i q^{\binom{i}{2}}x^{2ti}\\ &\quad - x^{1-4t^2}q^{1-4t^2}\sum_{i=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q)\end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned}\hat{\mathcal{H}}_t^{(m)}(qx;q) &= x^{-4t^2}q^{m-2t^2}\hat{\mathcal{H}}_t^{(m)}(x;q) + x^{-4t^2+m-t}q^{m-2t^2}(q)_\infty^3\sum_{i=1}^{2t}(-1)^i q^{\binom{i}{2}}x^{2ti}\\ &\quad + x^{-4t^2}q^{1-4t^2}\Theta(x;q)\sum_{i=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q).\end{aligned} \tag{3.2.2}$$

Proof. We first compute the sum

$$x^{4t^2-1}q^{2t^2-mt}\sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r,s)(-1)^{r+s}q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}\frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}} \tag{3.2.3}$$

in two ways. Expanding the numerator yields

$$x^{4t^2-1}q^{2t^2-m}\mathcal{H}_t^{(m)}(qx;q) - \sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r,s)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r+(r+1)(1-4t^2)+2t^2-m}}{1-xq^{r+1}}. \tag{3.2.4}$$

Taking $(r,s) \to (r-1, s+2t)$ in the second sum in (3.2.4) and using (2.1.17), (2.1.20) and (2.1.22) yields

$$\begin{aligned}&-\sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}}{1-xq^r}\\ &\qquad = -\mathcal{H}_t^{(m)}(x;q) - \frac{x^{m-t}(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^i q^{\binom{i}{2}}x^{2ti}.\end{aligned} \tag{3.2.5}$$

Alternatively, we use (3.1.6) to express (3.2.3) as

$$-q^{1-2t^2-m}\sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q). \tag{3.2.6}$$

Combining (3.2.4)–(3.2.6) gives us (3.2.1). Finally, (3.2.2) follows from (2.1.21) and (3.2.1). □

We can now prove our second main result.

Proof of Theorem 1.2.5. As $\hat{\mathcal{H}}_t^{(m)}(x) = \hat{\mathcal{H}}_t^{(m)}(x;q)$ does not have poles, we write

$$\hat{\mathcal{H}}_t^{(m)}(x) = \sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r x^{-r} \tag{3.2.7}$$

for all $x \in \mathbb{C}\setminus\{0\}$. Substituting (3.2.7) into (3.2.2), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}-r} a_r x^{-r} &= x^{-4t^2}q^{m-2t^2}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r x^{-r}\\ &\quad + x^{-4t^2+m-t}q^{m-2t^2}(q)_\infty^3\sum_{i=1}^{2t}(-1)^i q^{\binom{i}{2}}x^{2ti}\\ &\quad + x^{-4t^2}q^{1-4t^2}\Theta(x;q)\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q).\end{aligned} \tag{3.2.8}$$

Using (1.0.1) and (3.1.10), the last sum on the right-hand side of (3.2.8) can be written as

$$q^{1-4t^2}\sum_{r\in\mathbb{Z}}(-1)^r q^{\binom{r+1}{2}}\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+r(k-4t^2)+k} f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q)x^{-r}. \tag{3.2.9}$$

We now let $r \to r-4t^2$ in the first term on the right-hand side of (3.2.8), apply (3.2.9) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b_r' + c_r' \tag{3.2.10}$$

where

$$b_r' := q^{1-4t^2+\binom{r+1}{2}-\frac{r^2}{8t^2}-\frac{mr}{4t}+r(1-4t^2)}\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+k(r+1)} f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q)$$

and

$$c'_r := (-1)^{i+m+t}(q)_\infty^3 q^{m-2t^2+\binom{i}{2}-\frac{(4t^2-m+t-2ti)^2}{8t^2}-\frac{m}{4t^2}(4t^2-m+t-2ti)+(4t^2-m+t-2ti)}$$

if $r = 4t^2 - m + t - 2ti$, $1 \le i \le 2t$, and is 0 otherwise. Moreover, a similar computation as in (3.1.13) implies

$$\lim_{r\to\pm\infty} a_r = 0. \tag{3.2.11}$$

Now, observe that (3.2.10) is equivalent to

$$a_r - a_{r+4t^2} = b_r + c_r \tag{3.2.12}$$

where $b_r := -b'_{r+4t^2}$ and $c_r := -c'_{r+4t^2}$. We now claim that

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}. \tag{3.2.13}$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r$ and use (3.2.12) to obtain

$$\alpha_r = q^{-r-2t^2-m}\alpha_{r+4t^2} + q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} b_r + q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} c_r. \tag{3.2.14}$$

We will show

$$\alpha_r = q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

which clearly implies (3.2.13). Let

$$\tilde{a}_r := \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.2.12) and (3.2.14), respectively, via the same calculation as in (3.1.18) and (3.1.19) where we use (3.2.11) and $1 - t \le m \le t$. In addition,

$$\lim_{r\to\infty} (\alpha_r - \tilde{\alpha}_r) = 0. \tag{3.2.15}$$

Finally, we observe

$$\alpha_r - \tilde{\alpha}_r - q^{r+2t^2+m}\left(\alpha_{r-4t^2} - \tilde{\alpha}_{r-4t^2}\right) = 0$$

which in combination with (3.2.15) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$\begin{aligned}
\hat{\mathcal{H}}_t^{(m)}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l} x^{-r} \\
&= q^{1-m-8t^4} \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r}{2}+(4t^2-1)rl+4t^2(4t^2-1)\binom{l}{2}+r+(8t^4-m)l} \\
&\qquad \times \sum_{k=0}^{4t^2-2} (-1)^k q^{\binom{k+1-4t^2}{2}+kr+4t^2kl+4t^2k+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&= q^{1-m-2t^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&\qquad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r}{2}+(4t^2-1)rl+4t^2(4t^2-1)\binom{l}{2}+(k+1)r+(8t^4-m+4t^2k)l} x^{-r} \\
&= q^{1-m-2t^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&\qquad\qquad \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{8t^4-m+4t^2k}; q).
\end{aligned} \tag{3.2.16}$$

Thus, (1.2.12) follows from (1.2.11), (2.1.23), (2.1.26), (3.0.2) and (3.2.16). We now apply (2.1.1) and (2.1.20) to deduce that $f_{1,2t,1}$ is a modular form and (2.1.2) to obtain that $f_{1,4t^2-1,4t^2(4t^2-1)}$ is a false theta function. □

3.3 Proof of Theorem 1.2.7 $\left(\textbf{Modularity of } \mathcal{V}_t^{(m)}(x;q)\right)$

For the third case, we begin with the following result.

Proposition 3.3.1. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned}
\kappa_{t,m}(q^2x;q) &= -x^{1-2t} q^{3-4t^2-3t+4tm} \kappa_{t,m}(x;q) \\
&\quad - x^{-2t^2-t+2} q^{4-4t^2-4t+4tm} \frac{(q^2;q^2)_\infty^3}{\Theta(qx;q^2)} \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-2mi+ti} x^{ti} \\
&\quad - x^{1-2t^2} q^{3-6t^2} \sum_{k=0}^{2t^2-2} x^k q^{3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q)
\end{aligned} \tag{3.3.1}$$

and

$$\begin{aligned}\hat{\kappa}_{t,m}(q^2x;q) &= x^{-2t^2}q^{2-4t^2-3t+4tm}\hat{\kappa}_{t,m}(x;q)\\ &\quad + x^{-2t^2-t+1}q^{3-4t^2-4t+4tm}(q^2;q^2)_\infty^3\sum_{i=1}^{2t}(-1)^i q^{\binom{i+1}{2}-2mi+ti}x^{ti}\\ &\quad + x^{-2t^2}q^{2-6t^2}\Theta(qx;q^2)\sum_{k=0}^{2t^2-2}x^kq^{3k}f_{2,2t,1}(q^{2t+2-4t^2+2k},q^{2m};q).\end{aligned} \tag{3.3.2}$$

Proof. We first compute the sum

$$x^{2t^2-1}q^{-3+4t^2+3t-4tm}\sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r,s)(-1)^{r+s}q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}\frac{1-x^{1-2t^2}q^{(2r+3)(1-2t^2)}}{1-xq^{2r+3}} \tag{3.3.3}$$

in two ways. Expanding the numerator yields

$$\begin{aligned}&x^{2t^2-1}q^{-3+4t^2+3t-4tm}\kappa_{t,m}(q^2x;q)\\ &\qquad -\sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r,s)(-1)^{r+s}\frac{q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms-2t^2+3t+2r-4t^2r-4tm}}{1-xq^{2r+3}}.\end{aligned} \tag{3.3.4}$$

Taking $(r,s)\to(r-1,s+2t)$ in the second sum in (3.3.4) and using (2.1.17), (2.1.20) and (2.1.22) yields

$$\begin{aligned}&-\sum_{r,s\in\mathbb{Z}}\operatorname{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}}{1-xq^{2r+1}}\\ &\qquad = -\kappa_{t,m}(x;q)-\frac{(qx)^{1-t}(q^2;q^2)_\infty^3}{\Theta(qx;q^2)}\sum_{i=1}^{2t}(-1)^i q^{\binom{i+1}{2}-2mi+ti}x^{ti}.\end{aligned} \tag{3.3.5}$$

Alternatively, we use

$$\frac{1-x^{1-2t^2}q^{(2r+3)(1-2t^2)}}{1-xq^{2r+3}} = -x^{1-2t^2}q^{(2r+3)(1-2t^2)}\sum_{k=0}^{2t^2-2}x^kq^{(2r+3)k}$$

to express (3.3.3) as

$$-q^{-2t^2+3t-4tm}\sum_{k=0}^{2t^2-2}x^kq^{3k}f_{2,2t,1}(q^{2t+2-4t^2+2k},q^{2m};q). \tag{3.3.6}$$

Combining (3.3.4)–(3.3.6) gives us (3.3.1). Finally, (3.3.2) follows from (2.1.21) and (3.3.1). $\square$

We can now prove our third result.

Proof of Theorem 1.2.7. As $\hat{\kappa}_{t,m}(x) = \hat{\kappa}_{t,m}(x;q)$ does not have poles, we write

$$\hat{\kappa}_{t,m}(x) = \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2}r} a_r x^{-r} \tag{3.3.7}$$

for all $x \in \mathbb{C} \setminus \{0\}$. Substituting (3.3.7) into (3.3.2), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2}r-2r} a_r x^{-r} &= x^{-2t^2} q^{2-4t^2-3t+4tm} \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2}r} a_r x^{-r} \\ &+ x^{-2t^2-t+1} q^{3-4t^2-4t+4tm} (q^2;q^2)_\infty^3 \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-2mi+ti} x^{ti} \\ &+ x^{-2t^2} q^{2-6t^2} \Theta(qx;q^2) \sum_{k=0}^{2t^2-2} x^k q^{3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q).\end{aligned} \tag{3.3.8}$$

Using (1.0.1) and (3.1.10), the last sum on the right-hand side of (3.3.8) can be written as

$$q^{2-6t^2+4t^4} \sum_{r\in\mathbb{Z}} (-1)^r q^{r^2-4t^2r} \sum_{k=0}^{2t^2-2} (-1)^k q^{k^2+3k+2rk-4t^2k} f_{2,2t,1}(q^{2t+2+(2r-4t^2)k}, q^{2m}; q) x^{-r}. \tag{3.3.9}$$

We now let $r \to r - 2t^2$ in the first term on the right-hand side of (3.3.8), apply (3.3.9) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-2t^2} + b'_r + c'_r \tag{3.3.10}$$

where

$$\begin{aligned}b'_r := {} & q^{2+4t^4-6t^2+r^2-4t^2r-\frac{r^2}{2t^2}+\frac{2t^2-2+3t-4tm}{2t^2}r+2r} \\ & \times \sum_{k=0}^{2t^2-2} (-1)^k q^{k^2+3k+(2r-4t^2)k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q)\end{aligned}$$

and

$$c'_r := (-1)^{i+t+ti+1} (q^2;q^2)_\infty^3 q^{3-4t^2-4t+4tm+\binom{i+1}{2}-2mi+ti-\frac{(2t^2+t-1-ti)^2}{2t^2}+\frac{6t^2-2+3t-4tm}{2t^2}(2t^2+t-1-ti)}$$

if $r = 2t^2 + t - 1 - ti$, $1 \le i \le 2t$, and is 0 otherwise. Moreover, a similar computation as in (3.1.13) implies

$$\lim_{r\to\pm\infty} a_r = 0. \tag{3.3.11}$$

Now, observe that (3.3.10) is equivalent to

$$a_r - a_{r+2t^2} = b_r + c_r \tag{3.3.12}$$

where $b_r := -b'_{r+2t^2}$ and $c_r := -c'_{r+2t^2}$. We now claim that

$$a_r = \sum_{l\in\mathbb{Z}} \operatorname{sg}(r,l) b_{r+2t^2 l}. \tag{3.3.13}$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} a_r$ and use (3.3.12) to obtain

$$\alpha_r = q^{-2r-2+3t-4tm}\alpha_{r+2t^2} + q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} b_r + q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} c_r. \tag{3.3.14}$$

We will show

$$\alpha_r = q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} \sum_{l\in\mathbb{Z}} \operatorname{sg}(r,l) b_{r+2t^2 l}$$

which clearly implies (3.3.13). Let

$$\tilde{a}_r := \sum_{l\in\mathbb{Z}} \operatorname{sg}(r,l) b_{r+2t^2 l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} \tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.3.10) and (3.3.14), respectively, via the same calculation as in (3.1.18) and (3.1.19) with $r + 4t^2$ and $r + 4t^2 l$ replaced with $r + 2t^2$ and $r + 2t^2 l$, respectively, and (3.3.11). So,

$$\lim_{r\to\infty} (\alpha_r - \tilde{\alpha}_r) = 0. \tag{3.3.15}$$

Finally, we observe

$$\alpha_r - \tilde{\alpha}_r - q^{2r+2-3t+4tm}\left(\alpha_{r-2t^2} - \tilde{\alpha}_{r-2t^2}\right) = 0$$

which in combination with (3.3.15) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$\begin{aligned}
\hat{\kappa}_{t,m}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2} r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+2t^2 l} x^{-r} \\
&= q^{2+4t^4-6t^2} \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{(r+2t^2(l+1))^2 - 4t^2(r+2t^2(l+1)) - \frac{(r+2t^2(l+1))^2}{2t^2}} \\
&\quad \times q^{\frac{2t^2-2+3t-4tm}{2t^2}(r+2t^2(l+1)) + 2(r+2t^2(l+1))} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k+(2(r+2t^2(l+1))-4t^2)k + \frac{r^2}{2t^2}} \\
&\qquad\qquad \times q^{-\frac{2t^2-2+3t-4tm}{2t^2} r} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&= q^{-2t^2+3t-4tm} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&\qquad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{r^2+(4t^2-2)rl+2t^2l^2(2t^2-1)+2kr+(4t^2k+2t^2-2+3t-4tm)l} x^{-r} \\
&= q^{-2t^2+3t-4tm} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&\qquad \times f_{1,2t^2-1,2t^2(2t^2-1)}(x^{-1}q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q^2).
\end{aligned} \tag{3.3.16}$$

Thus, (1.2.17) follows from (1.2.16), (2.1.24), (2.1.27), (3.0.3) and (3.3.16). We now apply (2.1.1) and (2.1.20) to deduce that $f_{2,2t,1}$ is a modular form and (2.1.2) to obtain that $f_{1,2t^2-1,2t^2(2t^2-1)}$ is a false theta function. □

3.4 Proof of Theorem 1.2.9 $\left(\textbf{Modularity of } \mathcal{O}_t^{(m)}(x;q)\right)$

For our fourth case, we start with the following result.

Proposition 3.4.1. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned} p_{t,m}(qx;q) &= -x^{1-4t^2} q^{(m-1)(2t-1)} p_{t,m}(x;q) \\ &\quad - x^{1-4t^2-t-m+1} q^{(m-1)(2t-1)} \frac{(q)_\infty^3}{\Theta(x;q)} \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-i(t+m)} x^{2ti} \\ &\quad - x^{1-4t^2} q^{1-4t^2} \sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{t+m+1-4t^2+k}, q^{t+m}; q) \end{aligned} \tag{3.4.1}$$

and

$$\begin{aligned} \hat{p}_{t,m}(qx;q) &= x^{-4t^2} q^{(m-1)(2t-1)} \hat{p}_{t,m}(x;q) \\ &\quad + x^{-4t^2-t-m+1} q^{(m-1)(2t-1)} (q)_\infty^3 \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-i(t+m)} x^{2ti} \\ &\quad + x^{-4t^2} q^{1-4t^2} \Theta(x;q) \sum_{k=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{t+m+1-4t^2+k}, q^{t+m}; q). \end{aligned} \tag{3.4.2}$$

Proof. We first compute the sum

$$x^{4t^2-1} q^{-(m-1)(2t-1)} \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)(r+s)} \frac{1-x^{1-4t^2} q^{(r+1)(1-4t^2)}}{1-xq^{r+1}} \tag{3.4.3}$$

in two ways. Expanding the numerator yields

$$\begin{aligned} & x^{4t^2-1} q^{-(m-1)(2t-1)} p_{t,m}(qx;q) \\ & \quad - \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} \frac{q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m+1-4t^2)r+(t+m)s-(m-1)(2t-1)+1-4t^2}}{1-xq^{r+1}}. \end{aligned} \tag{3.4.4}$$

Taking $(r,s) \to (r-1, s+2t)$ in the second sum in (3.4.4) and using (2.1.17), (2.1.20) and (2.1.22) leads to

$$\begin{aligned}
&-\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)r+(t+m)s}}{1-xq^r}\\
&=-\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)r+(t+m)s}}{1-xq^r}+\frac{1}{1-x}\sum_{s\in\mathbb{Z}}(-1)^s q^{\binom{s}{2}+(t+m)s}\\
&-\sum_{i=1}^{2t}\sum_{r\in\mathbb{Z}}(-1)^{r-i}\frac{q^{\binom{r}{2}-2tri+\binom{i+1}{2}+(t+m)r-(t+m)i}}{1-xq^r}\\
&=-p_{t,m}(x;q)-\frac{x^{-t-m+1}(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^i q^{\binom{i+1}{2}-(t+m)i}x^{2ti}.
\end{aligned} \tag{3.4.5}$$

Alternatively, we use (3.1.6) to express (3.4.3) as

$$\begin{aligned}
&-q^{-(m-1)(2t-1)+1-4t^2}\sum_{k=0}^{4t^2-2}x^k q^k\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)(r+s)+r(1-4t^2+k)}\\
&=-q^{-(m-1)(2t-1)+1-4t^2}\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{t+m+1-4t^2+k},q^{t+m};q).
\end{aligned} \tag{3.4.6}$$

Combining (3.4.4)–(3.4.6) gives us (3.4.1). Finally, (3.4.2) follows from (2.1.21), (3.0.4) and (3.4.1). □

We can now prove our fourth result.

Proof of Theorem 1.2.9. As $\hat{p}_{t,m}(x)=\hat{p}_{t,m}(x;q)$ does not have poles, we write

$$\hat{p}_{t,m}(x)=\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{2t^2+(m-1)(2t-1)}{4t^2}r}a_r x^{-r} \tag{3.4.7}$$

for all $x\in\mathbb{C}\setminus\{0\}$. Substituting (3.4.7) into (3.4.2), we obtain

$$\begin{aligned}
\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{2t^2+(m-1)(2t-1)}{4t^2}r-r}a_r x^{-r} &= x^{-4t^2}q^{(m-1)(2t-1)}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{2t^2+(m-1)(2t-1)}{4t^2}r}a_r x^{-r}\\
&+x^{-4t^2+1-t-m}q^{(m-1)(2t-1)}(q)_\infty^3\sum_{i=1}^{2t}(-1)^i q^{\binom{i+1}{2}-(t+m)i}x^{2ti}\\
&+x^{-4t^2}q^{1-4t^2}\Theta(x;q)\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{t+m+1-4t^2+k},q^{t+m};q).
\end{aligned} \tag{3.4.8}$$

Using (1.0.1) and (3.1.10), the last sum on the right-hand side of (3.4.8) can be written as

$$q^{1-4t^2}\sum_{r\in\mathbb{Z}}(-1)^r q^{\binom{r+1}{2}}\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+r(k-4t^2)+k} f_{1,2t,1}(q^{t+m+1-4t^2+k},q^{t+m};q)x^{-r}. \tag{3.4.9}$$

We now let $r \to r-4t^2$ in the first term on the right-hand side of (3.4.8), apply (3.4.9) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b'_r + c'_r$$

where

$$b'_r := q^{1-4t^2+\binom{r+1}{2}-\frac{r^2}{8t^2}-\frac{2t^2+(m-1)(2t-1)-4t^2}{4t^2}r-4t^2r}\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+k(r+1)} \times f_{1,2t,1}(q^{t+m+1-4t^2+k},q^{t+m};q)$$

and

$$c'_r := (-1)^{i+1+t+m}(q)_\infty^3 \times q^{(m-1)(2t-1)+\binom{i+1}{2}-(t+m)i-\frac{(4t^2-1+t+m-2ti)^2}{8t^2}-\frac{2t^2+(m-1)(2t-1)-4t^2}{4t^2}(4t^2-1+t+m-2ti)}$$

if $r = t+m-1-2ti$, $1 \le i \le 2t$ and is 0 otherwise. As before, we have

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l)b_{r+4t^2l}$$

where $b_r := -b'_{r+4t^2}$ and so in total

$$\begin{aligned}
\hat{p}_{t,m}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{2t^2+(m-1)(2t-1)}{4t^2}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{2t^2+(m-1)(2t-1)}{4t^2}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2l} x^{-r} \\
&= q^{1-4t^2-(m-1)(2t-1)} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t+m+1-4t^2+k}, q^{t+m}; q) \\
&\quad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\frac{r^2}{2}+(4t^2-1)rl+2t^2l^2(4t^2-1)+(k+\frac{1}{2})r+(-(m-1)(2t-1)+4t^2k)l} x^{-r} \\
&= q^{1-4t^2-(m-1)(2t-1)} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t+m+1-4t^2}, q^{t+m}; q) \\
&\qquad \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{8t^4-2t^2-(m-1)(2t-1)+4t^2k}; q).
\end{aligned} \tag{3.4.10}$$

Thus, (1.2.22) follows from (1.2.21), (3.0.4) and (3.4.10). We now apply (2.1.1) and (2.1.20) to deduce that $f_{1,2t,1}$ is a modular form and (2.1.2) to obtain that $f_{1,4t^2-1,4t^2(4t^2-1)}$ is a false theta function.

3.5 Proof of Theorem 1.2.11 $\left(\textbf{Modularity of } V_t^{(m)}(x;q)\right)$

For our last case, we start with the following result.

Proposition 3.5.1. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned}
\Phi_t^{(m)}(q^{3t-1}x) &= (-1)^{t+1}x^{-1}q^{-m(3t-1)}\Phi_t^{(m)}(x) \\
&\quad + (-1)^t x^{-1} q^{-m(3t-1)} \sum_{i=0}^{3t-2} (-1)^i \frac{q^{\binom{i+1}{2}+mi}}{1-xq^i} \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) \\
&\quad + (-1)^{t+1} x^{3t-m-1} q^{\frac{(3t-1)(3t-2m-2)}{2}} \frac{(q)_\infty^3}{\Theta(x;q)} - x^{-1}q^{1-3t} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q)
\end{aligned} \tag{3.5.1}$$

and

$$\begin{aligned}\hat{\Phi}_t^{(m)}(q^{3t-1}x) &= x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)}\hat{\Phi}_t^{(m)}(x)\\&- x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)}\sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi}\Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)})\\&\qquad\times f_{1,1,1}(qx^{-1},q^{i+1};q)\\&+ x^{-m}q^{-m(3t-1)}(q)_\infty^3\\&+(-1)^t x^{-3t}q^{-\binom{3t}{2}}\Theta(x;q)f_{1,3t,3t(3t-1)}(q^m,-q^{\binom{3t}{2}+3t-1};q).\end{aligned} \tag{3.5.2}$$

Proof. We first compute the sum

$$\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^r q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}\frac{1-x^{-1}q^{-(r+3t-1)}}{1-xq^{r+3t-1}} \tag{3.5.3}$$

in two ways. Expanding the numerator yields

$$\Phi_t^{(m)}(q^{3t-1}x) - x^{-1}q^{-3t+1}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^r\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+mr+(\binom{3t}{2}+3t-1)s}}{1-xq^{r+3t-1}}. \tag{3.5.4}$$

Taking $(r,s)\to(r-(3t-1),s+1)$ in the second sum in (3.5.4) and using (2.1.17), (2.1.20) and (2.1.22) leads to

$$\begin{aligned}&(-1)^{t+1}q^{-(m-1)(3t-1)}\\&\quad\times\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r-(3t-1),s+1)(-1)^r\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}}{1-xq^r}\\&=(-1)^{t+1}q^{-(m-1)(3t-1)}\Phi_t^{(m)}(x)\\&\quad+(-1)^t q^{-(m-1)(3t-1)}\sum_{i=0}^{3t-2}(-1)^i\frac{q^{\binom{i}{2}+(m+1)i}}{1-xq^i}\Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)})\\&\quad+(-1)^{t+1}x^{3t-m}q^{\frac{(3t-1)(3t-2m-2)}{2}}\frac{(q)_\infty^3}{\Theta(x;q)}.\end{aligned} \tag{3.5.5}$$

Alternatively, we use

$$\frac{1-x^{-1}q^{-(r+3t-1)}}{1-xq^{r+3t-1}} = -x^{-1}q^{-(r+3t-1)} \tag{3.5.6}$$

to express (3.5.3) as

$$-x^{-1}q^{1-3t}f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q). \tag{3.5.7}$$

Combining (3.5.4)–(3.5.7) gives us (3.5.1). Finally, (3.5.2) follows from (2.1.21), (3.5.1) and

$$\frac{\Theta(x;q)}{1-xq^i} = f_{1,1,1}(qx^{-1}, q^{i+1}; q)$$

which holds for all $i \in \mathbb{Z}$. □

We can now prove our final result.

Proof of Theorem 1.2.11. As $\hat{\Phi}_t^{(m)}(x) = \hat{\Phi}_t^{(m)}(x;q)$ does not have poles, we write

$$\hat{\Phi}_t^{(m)}(x) = \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r} a_r x^{-r} \tag{3.5.8}$$

for all $x \in \mathbb{C} \setminus \{0\}$. Substituting (3.5.8) into (3.5.2), we obtain

$$\begin{aligned}
&\sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r - (3t-1)r} a_r x^{-r} \\
&\quad = \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r - \binom{3t-1}{2} - m(3t-1)} a_r x^{-r-3t} \\
&\quad - x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)} \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \\
&\qquad\qquad \times \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) f_{1,1,1}(qx^{-1}, q^{i+1}; q) \\
&\quad + x^{-m}q^{-m(3t-1)}(q)_\infty^3 + (-1)^t x^{-3t} q^{-\binom{3t}{2}} \Theta(x;q) f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q).
\end{aligned} \tag{3.5.9}$$

Using (1.0.1), the last sum on the right-hand side of (3.5.9) can be written as

$$f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q) \sum_{r\in\mathbb{Z}} (-1)^r q^{\binom{r}{2}-(3t-1)r} x^{-r}. \tag{3.5.10}$$

We now let $r \to r - 3t$ on the right-hand side of (3.5.9), apply (3.5.10) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b'_r + c'_r$$

where

$$b'_r := -q^{-\frac{(3t-1)r^2}{6t}+\frac{(m-1)(3t-1)}{3t}r+(3t-1)r-\binom{3t-1}{2}-m(3t-1)} \sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi}$$
$$\times \Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)}) \sum_{s\in\mathbb{Z}} \mathrm{sg}(r-3t,s)(-1)^{r+t+s} q^{\binom{r-3t+1}{2}+(r-3t)s+\binom{s+1}{2}+si}$$
$$+ (-1)^r q^{\binom{r}{2}-\frac{(3t-1)r^2}{6t}+\frac{(m-1)(3t-1)}{3t}r} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1};q)$$

and

$$c'_r := (q)_\infty^3 q^{-\frac{3t-1}{6t}m^2+\frac{(m-1)(3t-1)}{3t}m}$$

if $r=m$ and is 0 otherwise. As before, we have

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+3tl}$$

where $b_r := -b'_{r+3t}$ and so in total

$$\begin{aligned}
\hat{\Phi}_t^{(m)}(x) &= \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t}-\frac{(m-1)(3t-1)}{3t}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t}-\frac{(m-1)(3t-1)}{3t}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+3tl} x^{-r} \\
&= (-1)^{t+1} q^{(1-m)(1-3t)} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1};q) \\
&\qquad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^{r+l} q^{\binom{r+1}{2}+rl+3t\binom{l}{2}} ((-1)^{t+1} q^{3tm+1-m})^l x^{-r} \\
&+ \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)}) \\
&\times \sum_{r,l,s\in\mathbb{Z}} \mathrm{sg}(r,l)\,\mathrm{sg}(r+3tl,s)(-1)^{r+s+lt} \\
&\qquad \times q^{\binom{r+1}{2}+rl+3t\binom{l}{2}+rs+3tls+\binom{s+1}{2}+l(3mt+1-m)+is} x^{-r} \\
&= (-1)^{t+1} q^{(1-m)(1-3t)} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1};q) \\
&\qquad \times f_{1,1,3t}(x^{-1}q, (-1)^{t+1} q^{3tm+1-m};q) \\
&+ \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)}) \\
&\qquad \times \mathfrak{g}_{1,1,3t,1,3t,1}(x^{-1}q, (-1)^{t+1} q^{3mt+1-m}, q^{i+1};q)
\end{aligned} \tag{3.5.11}$$

where we have used (2.1.19) in the penultimate step. Thus, (1.2.28) follows from (1.2.26), (2.1.28), (3.0.5) and (3.5.11). Finally, $f_{1,3t,3t(3t-1)}$ is a mixed mock modular form by (2.1.1) and $f_{1,1,3t}$ is mixed false theta function by (2.1.2). □

Chapter 4

Recovering Partial Theta Identities

In [25], Hickerson and Mortenson use explicit formulas for certain classes of mixed mock $f_{a,b,c}$'s to obtain new proofs for mock theta identities. In a similar vein, we use our main results to recover the partial theta identities (1.0.6) – (1.0.9).

4.1 Recovering (1.0.6)

We set $t = m = 1$ in Theorem 1.2.3 to obtain

$$\mathcal{U}_1^{(1)}(x;q) = \frac{(1-x)}{\Theta(x;q)}\frac{q^{-1}}{(q)_\infty^2}\Big(-f_{1,2,1}(q^{-1},q;q)f_{1,3,12}(x^{-1}q,-q^8;q) + q^2 f_{1,2,1}(1,q;q)f_{1,3,12}(x^{-1}q^2,-q^{12};q) - q^5 f_{1,2,1}(q,q;q)f_{1,3,12}(x^{-1}q^3,-q^{16};q)\Big). \tag{4.1.1}$$

Using [25, Eq. (1.7)] and (2.1.21), one can show that $f_{1,2,1}(q^{-1},q;q) = -q(q)_\infty^2$, $f_{1,2,1}(1,q;q) = 0$ and $f_{1,2,1}(q,q;q) = (q)_\infty^2$ and so (4.1.1) becomes

$$\mathcal{U}_1^{(1)}(x;q) = \frac{(1-x)}{\Theta(x;q)}\Big(f_{1,3,12}(x^{-1}q,-q^8;q) - q^4 f_{1,3,12}(x^{-1}q^3,-q^{16};q)\Big). \tag{4.1.2}$$

By (1.0.1), (1.0.10), (2.1.21) , Theorem 2.1.2, the quintuple product identity

$$\sum_{k\in\mathbb{Z}} q^{\frac{k(3k-1)}{2}} x^{3k}(1-xq^k) = (q,x,q/x)_\infty(qx^2,qx^{-2};q^2)_\infty, \tag{4.1.3}$$

(4.1.2) and some simplifications, we have

$$\mathcal{U}_1^{(1)}(x;q) = (1-x)\Bigg(\frac{1}{2}\sum_{r\geq 0}(-1)^r x^{3r} q^{\frac{r(3r+1)}{2}}(1-x^2q^{2r+1}) - \frac{1}{2}\sum_{r<0}(-1)^r x^{3r} q^{\frac{r(3r+1)}{2}}(1-x^2q^{2r+1}) - \frac{1}{2\Theta(x;q)}\sum_{t=0}^{11}(-1)^t x^{-t} q^{\binom{t}{2}} \frac{\Theta(q^t;q^4)\Theta(q^{4+2t};q^8)}{(q^8;q^8)_\infty} \times \sum_{r\in\mathbb{Z}} \mathrm{sg}(r)(q^{-18+3t}x^{-12})^r q^{36\binom{r+1}{2}}\Bigg). \tag{4.1.4}$$

For t even, one of the theta functions in the numerator of the third term on the right-hand side of (4.1.4) is zero. Thus, it suffices to consider t odd. There are two steps. We first replace t with $4t+1$ and let $0 \leq t \leq 2$ in the third term on the right-hand side of (4.1.4). This eventually yields

$$\frac{x^{-1}}{2(x)_\infty(q/x)_\infty}\sum_{k\in\mathbb{Z}} \mathrm{sg}(k) x^{-4k} q^{\binom{2k+1}{2}}. \tag{4.1.5}$$

Next, we replace t with $4t+3$ and let $0 \leq t \leq 2$ in the third term on the right-hand side of (4.1.4). This gives

$$-\frac{x^{-3}}{2(x)_\infty(q/x)_\infty}\sum_{k\in\mathbb{Z}} \mathrm{sg}(k) x^{-4k} q^{\binom{2k+2}{2}}. \tag{4.1.6}$$

Combining (4.1.5) and (4.1.6), performing the shift $k \to -k-1$ and using $\mathrm{sg}(-k-1) = -\mathrm{sg}(k)$ leads to

$$\frac{x}{2(x)_\infty(q/x)_\infty}\sum_{k\in\mathbb{Z}} (-1)^k \mathrm{sg}(k) x^{2k} q^{\binom{k+1}{2}}. \tag{4.1.7}$$

Now, after inserting (4.1.7) into (4.1.4), using (1.0.10) and rearranging terms, we have

$$\mathcal{U}_1^{(1)}(x;q) = (1-x)\Bigg(\sum_{r\geq 0}(-1)^r x^{3r} q^{\frac{r(3r+1)}{2}}(1-x^2q^{2r+1}) - \frac{1}{2}\sum_{r\in\mathbb{Z}}(-1)^r x^{3r} q^{\frac{r(3r+1)}{2}}(1-x^2q^{2r+1}) + \frac{x}{(x)_\infty(q/x)_\infty}\sum_{k\geq 0}(-1)^k x^{2k} q^{\binom{k+1}{2}} - \frac{x}{2(x)_\infty(q/x)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k x^{2k} q^{\binom{k+1}{2}}\Bigg). \tag{4.1.8}$$

Applying (1.0.1) twice, then (2.1.21) and (4.1.3) to the second sum on the right-hand side of (4.1.8) implies

$$\sum_{r\in\mathbb{Z}}(-1)^r x^{3r} q^{\frac{r(3r+1)}{2}}(1-x^2q^{2r+1}) = x^{-1}\frac{(q)_\infty\Theta(x^2;q)}{\Theta(x;q)} \tag{4.1.9}$$

while (1.0.1) and (2.1.21) yield

$$\sum_{k\in\mathbb{Z}}(-1)^k x^{2k+1} q^{\binom{k+1}{2}} = -x^{-1}\Theta(x^2;q). \tag{4.1.10}$$

Thus, (1.0.6) follows from (4.1.8)–(4.1.10) and then replacing x with $-x$.

4.2 Recovering (1.0.7)

We first set $t = m = 1$ in Theorem 1.2.5, then use [25, Eq. (1.7)] and (2.1.21) to deduce that $f_{1,2,1}(q^{-2}, q, q) = -q^2(q)_\infty^2$, $f_{1,2,1}(q^{-1}, q; q) = -q(q)_\infty^2$ and $f_{1,2,1}(1, q; q) = 0$ in order to obtain

$$W_1^{(1)}(x;q) = -\frac{(1-x)}{(x)_\infty(q/x)_\infty} + \frac{(1-x)}{\Theta(x;q)}\Big(f_{1,3,12}(x^{-1}q, -q^7; q) + f_{1,3,12}(xq, -q^7; q) - qf_{1,3,12}(x^{-1}q^2, -q^{11}; q) - qf_{1,3,12}(xq^2, -q^{11}; q)\Big). \tag{4.2.1}$$

By (2.1.2), (4.1.3) and some simplifications, we have

$$\begin{aligned} f_{1,3,12}(xq, -q^7; q) - qf_{1,3,12}(xq^2, -q^{11}; q) &= \frac{\Theta(x;q)}{2}\Big(-x^{-1}\sum_{r\in\mathbb{Z}}\mathrm{sg}(r)(-1)^r x^{-3r} q^{3\binom{r+1}{2}-2r} \\ &\qquad - x^{-2}\sum_{r\in\mathbb{Z}}\mathrm{sg}(r)(-1)^r x^{-3r} q^{3\binom{r+1}{2}-r}\Big) \\ &\quad + \frac{1}{2}\sum_{t=0}^{11}(-1)^t x^t q^{\binom{t}{2}}\frac{\Theta(q^{t+1};q^4)\Theta(q^{6+2t};q^8)}{(q^8;q^8)_\infty}\sum_{r\in\mathbb{Z}}\mathrm{sg}(r)(q^{-15+3t}x^{12})^r q^{36\binom{r+1}{2}}. \end{aligned} \tag{4.2.2}$$

We now let $r \to -r-1$ in each of the first two terms on the right-hand side of (4.2.2), use $\mathrm{sg}(-r-1) = -\mathrm{sg}(r)$ and simplify to obtain

$$-\frac{\Theta(x;q)}{2}\sum_{r\in\mathbb{Z}}\mathrm{sg}(r)(-1)^r x^{3r+1} q^{\frac{(r+1)(3r+2)}{2}}(1+xq^{r+1}). \tag{4.2.3}$$

For t odd, one of the theta functions in the numerator of the third term on the right-hand side of (4.2.2) is zero. Thus, it suffices to consider t even. There are two steps. We first replace t with $4t$ and let $0 \le t \le 2$ in the third term on the right-hand side of (4.2.2). This eventually yields

$$\frac{(q)_\infty}{2} \sum_{k \in \mathbb{Z}} \mathrm{sg}(k) x^{4k} q^{\binom{2k+1}{2}}. \tag{4.2.4}$$

Next, we replace t with $4t+2$ and let $0 \le t \le 2$ in the third term on the right-hand side of (4.2.2). This gives

$$-\frac{(q)_\infty}{2} \sum_{k \in \mathbb{Z}} \mathrm{sg}(k) x^{4k+2} q^{\binom{2k+2}{2}}. \tag{4.2.5}$$

We now insert the sum of (4.2.4) and (4.2.5) along with (4.2.3) into (4.2.2) to obtain

$$\begin{aligned} & f_{1,3,12}(xq, -q^7; q) - q f_{1,3,12}(xq^2, -q^{11}; q) \\ &= -\frac{\Theta(x;q)}{2} \sum_{r \in \mathbb{Z}} \mathrm{sg}(r)(-1)^r x^{3r+1} q^{\frac{(r+1)(3r+2)}{2}} (1 + xq^{r+1}) + \frac{(q)_\infty}{2} \sum_{k \in \mathbb{Z}} \mathrm{sg}(k)(-1)^k x^{2k} q^{\binom{k+1}{2}}. \end{aligned} \tag{4.2.6}$$

A similar computation yields

$$\begin{aligned} & f_{1,3,12}(x^{-1}q, -q^7; q) - q f_{1,3,12}(x^{-1}q^2, -q^{11}; q) \\ &\qquad = \frac{\Theta(x;q)}{2} \sum_{r \in \mathbb{Z}} \mathrm{sg}(r)(-1)^r x^{3r} q^{\frac{r(3r-1)}{2}} (1 + xq^r) + \frac{(q)_\infty}{2} \sum_{k \in \mathbb{Z}} \mathrm{sg}(k)(-1)^k x^{-2k} q^{\binom{k+1}{2}}. \end{aligned} \tag{4.2.7}$$

After combining (4.2.6) and (4.2.7), (4.2.1), using (1.0.10) and rearranging terms, we have

$$\begin{aligned} W_1^{(1)}(x;q) = &-\frac{(1-x)}{(x)_\infty (q/x)_\infty} + (1-x)\Bigg(-\sum_{r \ge 0} (-1)^r x^{3r+1} q^{\frac{(r+1)(3r+2)}{2}} (1 + xq^{r+1}) \\ &+ \frac{1}{2} \sum_{r \in \mathbb{Z}} (-1)^r x^{3r+1} q^{\frac{(r+1)(3r+2)}{2}} (1 + xq^{r+1}) + \sum_{r \ge 0} (-1)^r x^{3r} q^{\frac{r(3r-1)}{2}} (1 + xq^r) \\ &- \frac{1}{2} \sum_{r \in \mathbb{Z}} (-1)^r x^{3r} q^{\frac{r(3r-1)}{2}} (1 + xq^r) + \frac{1+x^2}{(x)_\infty (q/x)_\infty} \sum_{k \ge 0} (-1)^k x^{2k} q^{\binom{k+1}{2}} \\ &- \frac{1+x^2}{2(x)_\infty (q/x)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k x^{2k} q^{\binom{k+1}{2}} \Bigg). \end{aligned} \tag{4.2.8}$$

Applying (1.0.1) twice, then (4.1.3) to each of the second and fourth sums and (1.0.1) to the sixth sum on the right-hand side of (4.2.8), then using (2.1.21) and cancelling the resulting theta functions leads to

$$W_1^{(1)}(x;q) = -\frac{(1-x)}{(x)_\infty(q/x)_\infty} + (1-x)\Bigg(-\sum_{r\geq 0}(-1)^r x^{3r+1} q^{\frac{(r+1)(3r+2)}{2}}(1+xq^{r+1}) \\ + \sum_{r\geq 0}(-1)^r x^{3r} q^{\frac{r(3r-1)}{2}}(1+xq^r) + \frac{1+x^2}{(x)_\infty(q/x)_\infty}\sum_{k\geq 0}(-1)^k x^{2k} q^{\binom{k+1}{2}}\Bigg). \tag{4.2.9}$$

Finally, we shift $r \to r-1$ in the first sum, remove the $r=0$ term from the second sum and the $k=0$ term from the third sum on the right-hand side of (4.2.9). In total, we have

$$W_1^{(1)}(x;q) = -\frac{(1-x)}{(x)_\infty(q/x)_\infty} + (1-x)\Bigg(1+x+(1+x^2)\sum_{r\geq 1}(-1)^r x^{3r-2} q^{\frac{r(3r-1)}{2}}(1+xq^r) \\ + \frac{1+x^2}{(x)_\infty(q/x)_\infty} + \frac{1+x^2}{(x)_\infty(q/x)_\infty}\sum_{k\geq 1}(-1)^k x^{2k} q^{\binom{k+1}{2}}\Bigg)$$

which is (1.0.7) after simplification.

4.3 Recovering (1.0.8)

We first set $t = m = 1$ in Theorem 1.2.7, then use [38, Corollary 3.11] to deduce that $f_{2,2,1}(1, q^2; q) = -q^3(q)_\infty(q^2;q^2)_\infty$ in order to obtain

$$\mathcal{V}_1^{(1)}(x;q) = -\frac{1}{1+x}\left(\frac{1}{\Theta(-q;q^2)}f_{1,1,2}(-q,-q;q^2) - \frac{1}{\Theta(qx;q^2)}f_{1,1,2}(x^{-1}q,-q;q^2)\right). \tag{4.3.1}$$

By (2.1.2) and some simplifications, we have

$$\begin{aligned} f_{1,1,2}(-q,-q;q^2) &= \frac{\Theta(-q;q^2)}{2}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{r^2-r} + \frac{\Theta(-q;q^4)}{2}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{\binom{r+1}{2}} \\ &= \Theta(-q;q^2) \end{aligned} \tag{4.3.2}$$

where in (4.3.2) the first sum is 2 while the second sum is 0 after letting $r \to -r-1$ and using that $\mathrm{sg}(-r-1) = -\mathrm{sg}(r)$. Similarly, one can check that

$$f_{1,1,2}(x^{-1}q, -q; q^2) = \frac{\Theta(qx;q^2)}{2} \sum_{r\in\mathbb{Z}} \mathrm{sg}(r)(-1)^r x^r q^{r^2-r} + \frac{\Theta(-q;q^4)}{2} \sum_{r\in\mathbb{Z}} \mathrm{sg}(r)(-1)^r x^{r+1} q^{\binom{r+1}{2}}. \tag{4.3.3}$$

By (1.0.10), (4.3.1)–(4.3.3) and rearranging terms, we have

$$\mathcal{V}_1^{(1)}(x;q) = -\frac{1}{1+x}\left(1 - \sum_{r\geq 0}(-1)^r x^r q^{r^2-r} + \frac{1}{2}\sum_{r\in\mathbb{Z}}(-1)^r x^r q^{r^2-r} - \frac{\Theta(-q;q^4)}{\Theta(qx;q^2)} \sum_{r\geq 0}(-1)^r x^{r+1} q^{\binom{r+1}{2}} + \frac{\Theta(-q;q^4)}{2\Theta(qx;q^2)} \sum_{r\in\mathbb{Z}}(-1)^r x^{r+1} q^{\binom{r+1}{2}}\right). \tag{4.3.4}$$

Finally, we apply (1.0.1) to the second and fourth sums, cancel the resulting theta functions, remove the $r = 0$ term and then perform the shift $r \to r+1$ in the first sum on the right-hand side of (4.3.4). In total,

$$\mathcal{V}_1^{(1)}(x;q) = -\frac{1}{1+x}\left(\sum_{r\geq 0}(-1)^{r+1} x^{r+1} q^{r(r+1)} - \frac{\Theta(-q;q^4)}{2\Theta(qx;q^2)} \sum_{r\geq 0}(-1)^r x^{r+1} q^{\binom{r+1}{2}}\right)$$

which is (1.0.8) after simplification.

4.4 Recovering (1.0.9)

We first set $t = m = 1$ in Theorem 1.2.9, then use [25, Eq. (1.7)] and (2.1.21) to deduce that $f_{1,2,1}(q^{-2}, q^4; q^2) = 0$, $f_{1,2,1}(1, q^4; q^2) = q^2(q^2;q^2)_\infty^2$ and $f_{1,2,1}(q^2, q^4; q^2) = (q^2;q^2)_\infty^2$ in order to obtain

$$\mathcal{O}_1^{(1)}(x;q) = \frac{q}{\Theta(xq;q^2)}\Big(f_{1,3,12}(x^{-1}q^3, -q^{20}; q^2) - q^4 f_{1,3,12}(x^{-1}q^5, -q^{28}; q^2)\Big). \tag{4.4.1}$$

By (1.0.1), (2.1.2), (2.1.21) and (4.1.3) and some simplifications, we have

$$\begin{aligned} &f_{1,3,12}(x^{-1}q^3, -q^{20}; q^2) - q^4 f_{1,3,12}(x^{-1}q^5, -q^{28}; q^2) \\ &\quad = -\frac{q^{-1}\Theta(xq;q^2)}{2} \sum_{r\in\mathbb{Z}} \mathrm{sg}(r)(-1)^r x^{3r+1} q^{3r^2+2r}(1+xq^{2r+1}) \\ &\quad + \frac{1}{2}\sum_{t=0}^{11}(-1)^t x^{-t} q^{t^2+2t} \frac{\Theta(q^{2t+4};q^8)\Theta(q^{16+4t};q^{16})}{(q^{16};q^{16})_\infty} \sum_{r\in\mathbb{Z}} \mathrm{sg}(r) x^{-12r} q^{6tr+36r^2}. \end{aligned} \tag{4.4.2}$$

For t even, one of the theta functions in the numerator of the second term on the right-hand side of (4.4.2) is zero. Thus, it suffices to consider t odd. There are two steps. We first replace t with $4t+1$ and let $0 \le t \le 2$ in the second term on the right-hand side of (4.4.2). This eventually yields

$$\frac{q^{-1}x^{-1}(q^2;q^2)_\infty}{2} \sum_{k\in\mathbb{Z}} \mathrm{sg}(k) x^{-4k} q^{2\binom{2k+1}{2}}. \tag{4.4.3}$$

Next, we replace t with $4t+3$ and let $0 \le t \le 2$ in the third term on the right-hand side of (4.4.2). This gives

$$-\frac{q^{-1}x^{-1}(q^2;q^2)_\infty}{2} \sum_{k\in\mathbb{Z}} \mathrm{sg}(k) x^{-4k-2} q^{2\binom{2k+2}{2}}. \tag{4.4.4}$$

We now combine (4.4.3) and (4.4.4), perform the shift $k \to -k-1$ and use that $\mathrm{sg}(-k-1) = -\mathrm{sg}(k)$ to obtain

$$\frac{q^{-1}(q^2;q^2)_\infty}{2} \sum_{k\in\mathbb{Z}} \mathrm{sg}(k)(-1)^k x^{2k+1} q^{2\binom{k+1}{2}}. \tag{4.4.5}$$

Now, after inserting (4.4.5) into (4.4.2), (4.4.1), using (1.0.10) and rearranging terms, we have

$$\begin{aligned} \mathcal{O}_1^{(1)}(x;q) = -\frac{1}{2}\Bigg(& 2\sum_{r\ge 0}(-1)^r x^{3r+1} q^{3r^2+2r}(1+xq^{2r+1}) \\ & - \sum_{r\in\mathbb{Z}}(-1)^r x^{3r+1} q^{3r^2+2r}(1+xq^{2r+1}) \Bigg) \\ + \frac{(q^2;q^2)_\infty}{2\Theta(xq;q^2)} & \left(2\sum_{k\ge 0}(-1)^k x^{2k+1} q^{2\binom{k+1}{2}} - \sum_{k\in\mathbb{Z}}(-1)^k x^{2k+1} q^{2\binom{k+1}{2}} \right). \end{aligned} \tag{4.4.6}$$

Note that two applications of (1.0.1) followed by (4.1.3) implies

$$\sum_{r\in\mathbb{Z}}(-1)^r x^{3r+1}q^{3r^2+2r}(1+xq^{2r+1}) = x\frac{\Theta(-xq;q^2)\Theta(x^2q^4;q^4)}{(q^4;q^4)_\infty} \tag{4.4.7}$$

while (1.0.1) yields

$$\sum_{k\in\mathbb{Z}}(-1)^k x^{2k+1}q^{2\binom{k+1}{2}} = x\Theta(x^2q^2;q^2). \tag{4.4.8}$$

Thus, (1.0.9) follows from (4.4.6)–(4.4.8), cancelling the theta functions and then replacing x with $-x$.

Chapter 5

Future Directions

5.1 Beyond the Main Theorems

There are numerous directions for further study. For example, it would be worthwhile to find a q-multisum expression for $\mathcal{U}_t^{(m)}(x;q)$ (and the other cases in Theorems 1.2.5, 1.2.7, 1.2.9 and 1.2.11), akin to (1.1.2) for the *strongly unimodal sequence* generating function, in order to discover a combinatorial interpretation for their coefficients or a potential connection to the coloured Jones polynomial for some family of knots. A possible starting point is the fact that (1.2.4) is an application of the techniques in [32] combined with appropriately chosen Bailey pairs. A strategy would be to find two-parameter generalisations of the Bailey pair identities of [32, Theorems 1.1 and 1.2]. Another strategy would be to use techniques of Warnaar analogous to [46] where he generalises other partial theta identities from the lost notebook. Warnaar generalises (1.0.6) in Theorem 1.4 of [46]. We provide an alternative generalisation using Theorem 1.2.3. For $t, m \in \mathbb{Z}$ such that $-t \leq m \leq 3t-2$, one can show

$$\begin{aligned}
\mathcal{U}_t^{(m)}(x;q) = &-\frac{1}{2}\frac{1-x}{(q)_\infty^2}q^{1-3t^2-\frac{t-m}{2}+tm}\sum_{k=0}^{4t^2-2}x^kq^kf_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)\\
&\times\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)(-1)^rx^{(4t^2-1)r}q^{(4t^2-1)\binom{r}{2}+r(k+t^2+tm-\frac{t-m}{2})}\\
&+\frac{1}{2}\frac{1-x}{\Theta(x;q)}\frac{q^{1-3t^2-\frac{t-m}{2}+tm}}{(q)_\infty^2}\sum_{k=0}^{4t^2-2}(-1)^{k+1}q^{\binom{k+1}{2}+k}f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)\\
&\times\sum_{l=0}^{4t^2-1}(-1)^lx^{-l}q^{kl+\binom{l+1}{2}}\Theta(-q^{(4t^2-1)l+4t^2-t^2+tm-\frac{t-m}{2}+8t^4};q^{4t^2(4t^2-1)})\\
&\times\sum_{R\in\mathbb{Z}}\operatorname{sg}(R)x^{-4t^2R}q^{2t^2R^2+R(tm-\frac{t-m}{2}+l+t^2)}.
\end{aligned}$$

Similarly, we can generalise (1.0.9) using Theorem 1.2.9. For $t,m\in\mathbb{Z}$ such that $1-t\le m\le t$, one can show

$$\begin{aligned}
\mathcal{O}_t^{(m)}(x;q) = &-\frac{1}{2}\frac{q^{3-8t^2-2(m-1)(2t-1)}}{(q^2;q^2)_\infty^2}\sum_{k=0}^{4t^2-2}q^{3k}x^kf_{1,2t,1}(q^{2t+2m+2-8t^2+2k},q^{2t+2m};q^2)\\
&\times\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)(-1)^rx^{(4t^2-1)r}q^{(4t^2-1)r^2+(2k-2(m-1)(2t-1))r}\\
&+\frac{1}{2}\frac{q^{3-8t^2-2(m-1)(2t-1)}}{\Theta(xq;q^2)(q^2;q^2)_\infty^2}\sum_{k=0}^{4t^2-2}(-1)^{k+1}q^{k^2+3k}f_{1,2t,1}(q^{2t+2m+2-8t^2+2k},q^{2t+2m};q^2)\\
&\times\sum_{l=0}^{4t^2-1}(-1)^lx^{-l}q^{l^2+2kl}\Theta(-q^{2(4t^2-1)l+4t^2(4t^2-1)+8t^2k-2(m-1)(2t-1)}:q^{8t^2(4t^2-1)})\\
&\times\sum_{R\in\mathbb{Z}}\operatorname{sg}(R)x^{-4t^2R}q^{4t^2R^2+R(2l+2(m-1)(2t-1))}.
\end{aligned}$$

These are analogous to the right-hand sides of Theorems 1.1–1.4 of Warnaar in [46]. Recovering the respective left-hand sides of these Warnaar-type identities would yield q-multisum expressions for $\mathcal{U}_t^{(m)}$ and $\mathcal{O}_t^{(m)}$.

Although $\mathcal{U}_t^{(m)}(x;q)$ may not have monotonic coefficients when x is some rational power of q, we demonstrate one instance of $\mathcal{U}_t^{(m)}(x;q)$ with a neat combinatorial interpretation. For $t=2$ and $m=0$, we have that

$$\mathcal{U}_2^{(0)}(1;q) = 1+q+2q^2+4q^3+8q^4+15q^5+27q^6+47q^7+79q^8+\cdots$$

is the generating function $\sum_{n\geq 0} \mathrm{un}(n)q^n$ where $\mathrm{un}(n)$ is the number of unimodal sequences of weight n such that sequences with a repeated peak are not counted multiple times. In comparison to (1.2.2), $\mathrm{un}(4) = 8$ and the sequences of weight 4 are

$$(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1).$$

5.2 Hecke-Appell Sums

For convenience, recall the Hecke-Appell expansions for the *unimodal sequence* generating function,

$$\mathcal{U}(x;q) = \frac{(1-x)}{(q)_\infty^2}\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2rs+\frac{s^2}{2}+\frac{3}{2}r+\frac{1}{2}s}}{1-xq^r}, \tag{5.2.1}$$

and the *strongly unimodal sequence* generating function

$$\begin{aligned} U_1^{(1)}(-x;q) &= -q^{-\frac{5}{8}}\frac{(qx)_\infty(x^{-1}q)_\infty}{(q)_\infty^2} \\ &\times \left(\sum_{\substack{r,s\geq 0\\ r\not\equiv s \pmod 2}} - \sum_{\substack{r,s<0\\ r\not\equiv s \pmod 2}}\right)\frac{(-1)^{\frac{r-s-1}{2}}q^{\frac{1}{8}r^2+\frac{7}{4}rs+\frac{1}{8}s^2+\frac{3}{2}r+\frac{1}{2}s}}{1-xq^{\frac{r+s+1}{2}}} \end{aligned} \tag{5.2.2}$$

One can study generalisations of (5.2.1) and (5.2.2), namely

$$\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{Q(r,s)+P(r,s)}}{1-xq^r} \tag{5.2.3}$$

and

$$\sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{Q(r,s)+P(r,s)}}{1-xq^{r+s}} \tag{5.2.4}$$

where $Q(r,s)$ is a quadratic form and $P(r,s)$ is a linear form, and the *numerator sum*

$$\sum_{r,s\in\mathbb{Z}} sg(r,s)(-1)^{r+s}q^{Q(r,s)+P(r,s)} \tag{5.2.5}$$

is a modular form. By the results in this thesis and [35], the first type should correspond to mixed false theta functions and the second type to mixed mock theta functions.

Consider a specialisation of (5.2.3) and (5.2.4) at $x = q^{-c}$ for some $c \in \mathbb{Z}$. Observe that (5.2.3) features a pole when $r = c$, for all $s \in \mathbb{Z}$. However, (5.2.4) features a pole when $r + s = c$, i.e., finitely many terms in the sum. This leads to the question of whether or not the behaviour of these functions at the poles characterises their modular properties.

5.3 Quantum Modular Forms

In [50], Zagier introduced the notion of a quantum modular form. Although the canonical definition is in flux, we define a *quantum modular form* of weight $k \in \frac{1}{2}\mathbb{Z}$ as a function $f : \mathbb{Q} \to \mathbb{C}$ such that

$$f(\tau) - (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

extends to a real-analytic function on $\mathbb{P}^1 \setminus S_\gamma$, where S_γ is a finite set for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Given that false theta functions are examples of quantum modular forms [24, Section 4.4], it would be highly desirable to investigate whether Theorems 1.2.3, 1.2.5, 1.2.7, 1.2.9 and 1.2.11 lead to the construction of new families of quantum Jacobi forms in the spirit of [23].

5.4 Higher Dimensional Analogues of $f_{a,b,c}(x, y; q)$

Recall the classification of Hecke-type double sums using the discriminant D in Remark 2.1.7. D is obviously related to the discriminant of the quadratic form in the exponent of $f_{a,b,c}$. It would be interesting to determine a similar characterisation for higher-dimensional Hecke-type sums of the form

$$\sum_{\substack{r_1, r_2, \ldots, r_n \in \mathbb{Z} \\ \mathrm{sg}(r_1) = \mathrm{sg}(r_2) = \cdots = \mathrm{sg}(r_n)}} (-1)^{r_1 + r_2 + \cdots + r_n} q^{Q(r_1, r_2, \ldots, r_n)} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}.$$

Is there a higher-dimensional analogue to the classification theorem of $f_{a,b,c}$ using the discriminant of an arbitrary quadratic form $Q(r_1, r_2, \ldots, r_n)$?

Bibliography

[1] K. Allen, R. Osburn, *Unimodal sequences and mixed false theta functions*, Adv. Math. **473** (2025), Paper No. 110293.

[2] G. E. Andrews, *An introduction to Ramanujan's "lost" notebook*, American Mathematical Society, Providence, RI, 2001, 165–184.

[3] G. E. Andrews, *Ramanujan's "lost" notebook. I. Partial θ-functions*, Adv. in Math. **41** (1981), no. 2, 137–172.

[4] G. E. Andrews, *Ramanujan's "lost" notebook. IV. Stacks and alternating parity in partitions*, Adv. in Math. **53** (1984), no. 1, 55–74.

[5] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part I*, Springer, New York, 2005.

[6] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part II*, Springer, New York, 2009.

[7] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part III*, Springer, New York, 2012.

[8] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part IV*, Springer, New York, 2013.

[9] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part V*, Springer, New York, 2018.

[10] G. E. Andrews, D. Hickerson, *Ramanujan's "lost" notebook. VII. The sixth order mock theta functions*, Adv. Math. **89** (1991), no. 1, 60–105.

[11] B. C. Berndt, *Ramanujan's Lost Notebook: How was it found?*, TheLesserKnownMath, https://www.youtube.com/watch?v=_EawVMk8ibE, (2023).

[12] N. E. Borozenets, *The mixed mock modularity of a new U-type function related to the Andrews-Gordon identities*, Hardy-Ramanujan J. **45** (2022), 108–129.

[13] P. Brändén, *Unimodality, log-concavity, real-rootedness and beyond*, Handbook of enumerative combinatorics, 437–483, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015.

[14] F. Brenti, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update*, Jerusalem combinatorics '93, 71–89, Contemp. Math., **178**, Amer. Math. Soc., Providence, RI, 1994.

[15] K. Bringmann, J. Lovejoy, *Odd unimodal sequences*, preprint available at https://arxiv.org/abs/2308.11556

[16] K. Bringmann, L. Rolen and S. Zwegers, *On the modularity of certain functions from the Gromov-Witten theory of elliptic orbifolds*, R. Soc. Open Sci. **2**, November (2015), 150310, 12 pp.

[17] K. Bringmann, C. Nazaroglu, *A framework for modular properties of false theta functions*, Res. Math. Sci. **6** (2019), no. 3, Paper No. 30, 23 pp.

[18] J. Bruinier, J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.

[19] J. Bruinier, G. Geer, G. Harder, D. Zagier, *The 1-2-3 of Modular Forms*, Universitext, Springer-Verlag, Berlin (2008).

[20] J. Bryson, K. Ono, S. Pitman and R. Rhoades, *Unimodal sequences and quantum and mock modular forms*, Proc. Natl. Acad. Sci. USA **109** (2012), no. 40, 16063–16067.

[21] S. Corteel, J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1623–1635.

[22] A. Dabholkar, S. Murthy and D. Zagier, *Quantum black holes, wall crossing, and mock modular forms*, preprint available at https://arxiv.org/abs/1208.4074

[23] A. Folsom, *Quantum Jacobi forms in number theory, topology, and mathematical physics*, Res. Math. Sci. **6** (2019), no. 3, Paper No. 25, 34 pp.

[24] A. Goswami, R. Osburn, *Quantum modularity of partial theta series with periodic coefficients*, Forum Math. **33** (2021), no. 2, 451–463.

[25] D. R. Hickerson, E. T. Mortenson, *Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 2, 382–422.

[26] K. Hikami, J. Lovejoy, *Torus knots and quantum modular forms*, Res. Math. Sci. **2** (2015), Art. 2, 15 pp.

[27] K. Hikami, J. Lovejoy, *Hecke-type formulas for families of unified Witten-Reshetikhin-Turaev invariants*, Commun. Number Theory Phys. **11** (2017), no. 2, 249–272.

[28] B. Kim, J. Lovejoy, *The rank of a unimodal sequence and a partial theta identity of Ramanujan*, Int. J. Number Theory **10** (2014), no. 4, 1081–1098.

[29] B. Kim, J. Lovejoy, *Ramanujan-type partial theta identities and rank differences for special unimodal sequences*, Ann Comb. **19** (2015), no. 4, 705–733.

[30] B. Kim, J. Lovejoy, *Partial indefinite theta identities*, J. Aust. Math. Soc. **102** (2017), no. 2, 255–289.

[31] G. Korpas, J. Manschot, G. Moore and I. Nidaiev, *Mocking the u-plane integral*, Res. Math. Sci. **8** (2021), no. 3, Paper No. 43, 42 pp.

[32] J. Lovejoy, *Ramanujan-type partial theta identities and conjugate Bailey pairs*, Ramanujan J. **29** (2012), no. 1-3, 51–67.

[33] J. Lovejoy, R. Osburn, *Mixed mock modular q-series*, J. Indian Math. Soc. (N.S.) 2013, Special volume to commemorate the 125th birth anniversary of Srinivasa Ramanujan, 45–61.

[34] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. **3** (2003), 537–556.

[35] E. T. Mortenson, S. Zwegers, *The mixed mock modularity of certain duals of generalized quantum modular forms of Hikami and Lovejoy*, Adv. Math. **418** (2023), Paper No. 108944, 27 pp.

[36] E. T. Mortenson, *A heuristic guide to evaluating triple-sums*, Hardy-Ramanujan J. **43** (2020), 99–121.

[37] E. T. Mortenson, *A general formula for Hecke-type false theta functions*, Algebra i Analiz **36** (2024), no. 1, 195–203.

[38] E. T. Mortenson, O. Postnova and D. Solovyev, *On string functions and double-sum formulas*, Res. Math. Sci. **10** (2023), no. 2, Paper No. 15, 23 pp.

[39] A. Polishchuk, *Indefinite theta series of signature* $(1,1)$ *from the point of view of homological mirror symmetry*, Adv. Math. **196** (2005), no. 1, 1–51.

[40] S. Ramanujan, *The lost notebook and other unpublished papers*, Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, (1988).

[41] M. Ram Murty, *Unimodal sequences: from Isaac Newton to June Huh*, Int. J. Number Theory **20** (2024), no. 10, 2453–2476.

[42] L. J. Rogers, *On two theorems of combinatorial analysis and some allied identities*, Proc. Lond. Math. Soc. (2) **16** (1917), no. 1, 315–336.

[43] A. Semikhatov, A. Taormina and I. Tipunin, *Higher-level Appell functions, modular transformations, and characters*, Comm. Math. Phys. **255** (2005), no. 2, 469–512.

[44] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Graph theory and its applications: East and West (Jinan, 1986), 500–535, Ann. New York Acad. Sci., **576**, New York Acad. Sci., New York, 1989.

[45] M. Viazovska, *The sphere packing problem in dimension 8*, Ann. of Math. (2) **185** (2017), no. 3, 991–1015.

[46] S. O. Warnaar, *Partial theta functions. I. Beyond the lost notebook*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 363–395.

[47] G. N. Watson, *The final problem: An account of the mock theta functions*, American Mathematical Society, Providence, RI, 2001, 325–347.

[48] A. Wiles, *Modular Elliptic Curves and Fermat's Last Theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

[49] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Séminaire Bourbaki. Vol. 2007/2008. Astérisque **326** (2009), Exp. No. 986, vii–viii, 143–164 (2010).

[50] D. Zagier, *Quantum modular forms*, Quanta of maths, 659–675, Clay Math. Proc., **11**, Amer. Math. Soc., Providence, RI (2010).

[51] S. Zwegers, *Mock theta functions*, Ph.D. thesis, Universiteit Utrecht (2002).

[52] S. Zwegers, *On two fifth order mock theta functions*, Ramanujan J. **20** (2009), no. 2, 207–214.